Duality of fully fuzzy linear system

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Abstract: According to fuzzy arithmetic, dual fully fuzzy linear system can not be replaced by a fully fuzzy linear system. In this paper we investigate the existence of duality fully fuzzy linear equation system. We propose a method to solve this system. Some numerical examples are given.

Key words: Fully fuzzy linear system- Duality- nonnegative matrix.

INTRODUCTION

The topic of fuzzy linear systems, which attracted increasing interest for some time, in particular in relation to fuzzy neural network, has been rapidly grown in recent years (Buckley Qu 1990: Friedman et al., 1998).

A general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector, first proposed by Friedman et al., (1998). Dehghan (2006) extended some iterative methods on the same system and discussed (Dehghan Hashemi 2006) the case in which all parameters in a fuzzy linear system are fuzzy numbers, which is called a fully fuzzy linear system (FFLS).

As claimed by Skalna et al. (2007), it is necessary to analyze sensitivity of the computed solutions of fuzzy linear systems to the coefficient and the right hand vector.

Friedman et al. (2000). investigated duality in fuzzy linear systems. Tow necessary and sufficient conditions for the solution existence were given.

Some works were followed and modified by others (Ezzati 2008: Allahviranloo and Salahshour 2010).

In this paper, we solve fully fuzzy linear system $\mathcal{A} \otimes \mathcal{X} = \mathcal{B} \otimes \mathcal{X} + \mathcal{C}$ Where $\mathcal{A} = (\tilde{a}_{ij})_{nm}$, $\mathcal{B} = (\tilde{b}_{ij})_{nm}$, $1 \leq i, j \leq n$, be fuzzy number matrices and $\mathcal{C}$ a fuzzy number vector.

Usually, there is no inverse element for an arbitrary fuzzy number $\tilde{u}$, i.e. there exists no element $\tilde{v}$ such that $\tilde{u} \oplus \tilde{v} = 0$.

Actually, for all non-crisp fuzzy number $\tilde{u}$ we have $\tilde{u} \oplus (-\tilde{u}) \neq 0$.

Therefore, the fully fuzzy linear system $\mathcal{A} \otimes \mathcal{X} = \mathcal{B} \otimes \mathcal{X} + \mathcal{C}$ cannot be equivalently replace by the fuzzy linear equation system $(\mathcal{A} - \mathcal{B}) \otimes \mathcal{X} = \mathcal{C}$

Which had been investigated. In the sequel, we will call the fully fuzzy linear system $\mathcal{A} \otimes \mathcal{X} = \mathcal{B} \otimes \mathcal{X} + \mathcal{C}$ a dual fully fuzzy linear system (DFFLS).

Preliminaries:

Definition 2.1. Let $X$ denote a universal set. Then a fuzzy subset $\tilde{A}$ of $X$ is defined by its membership function

$\mu_{\tilde{A}}: X \rightarrow [0,1]$,

Which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$, to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$ at $x$ shows the grade of membership of $x$ in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element $x$ and grade $\mu_{\tilde{A}}(x)$ and is often written

$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$.

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Definition 2.2. The r-level set of a fuzzy set $\tilde{A}$ is defined as an ordinary set $[\tilde{A}]_r$ for which the degree of its membership function exceeds the level r

$$[\tilde{A}]_r = \{x | \mu_\tilde{A}(x) \geq r, r \in (0,1]\}.$$ 

Definition 2.3. A fuzzy set $\tilde{A}$ in $\mathbb{R}^n$ is said to be a convex fuzzy set if and only if its r-level sets are convex.

Definition 2.4. A fuzzy set $\tilde{A}$ in $\mathbb{R}^n$ is said to be normal if there exist $x \in \mathbb{R}^n$ such that $\mu_\tilde{A}(x) = 1$.

Definition 2.5. A fuzzy number is a convex normalized fuzzy set of the real line $\mathbb{R}$ whose membership function is piecewise continuous.

Definition 2.6. A fuzzy number $m_\tilde{A}$ is called positive (negative), denoted by $m_\tilde{A} > 0$ ($m_\tilde{A} < 0$), if its membership function $\mu_{m_\tilde{A}}(x) = \begin{cases} 0 & \text{if } x < m, \alpha > 0 \\ m_\tilde{A} - x & \text{if } m \leq x, \alpha > 0 \\ -m_\tilde{A} + x & \text{if } x \geq m, \beta > 0 \end{cases}$

Where $m$ is the mean value of $\tilde{m}$ and $\alpha$ and $\beta$ are left and right spreads, respectively, and a function $L(x)$, the left shape function, satisfying

1. $L(x) = L(-x)$,
2. $L(0) = 1$ and $L(1) = 0$,
3. $L(x)$ is non-increasing on $[0, \infty)$.
4. Naturally, a right shape function, $R(x)$ is similarly defined as $L(x)$.
5. Remark 2.9. A fuzzy number could be neither positive, nor negative.

Definition 2.10. Two LR fuzzy numbers $\tilde{m} = (m, \alpha, \beta)_{LR}$ and $\tilde{n} = (n, \gamma, \delta)_{LR}$ are said to be equal, if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.

Definition 2.11. A matrix $A_\tilde{A}$ is called a fuzzy matrix, if each element of $A_\tilde{A}$ is a fuzzy number and will be positive (negative) if each element of $A_\tilde{A}$ be positive (negative).

Definition 2.12. Consider the $n \times n$ fully fuzzy linear system of equation:
The matrix form of the above equation is \[ \tilde{A} \otimes \tilde{X} = \tilde{B} \] where \( \tilde{A} = (A, M, N) \) and \( \tilde{X} = (X, Y, Z) \) and \( \tilde{B} = (b, h, g) \).

**Dual Fully Fuzzy Linear System (DFFLS):**

Finding positive solution of DFFLS \( \tilde{A} \otimes \tilde{X} = \tilde{B} \) where \( \tilde{A} = (A, M, N) \), \( \tilde{B} = (B, M', N') \) and \( \tilde{C} = (b, h, g) \) are positive, leads to

\[ (AX + MY + M'X, AZ + NX) = (Bx, By + M'X, AZ + N'X) + (b, h, g) \]

So we have

\[ (A - B)X = b, \quad (A - B)Y + (M - M')X = h \quad \text{and} \quad (A - B)Z + (N - N')X = g. \]

By assuming \( A - B \) is nonsingular crisp matrix, then

\[ X = (A - B)^{-1}b, \quad Y = (A - B)^{-1}h - (A - B)^{-1}(M - M')X \quad \text{and} \quad Z = (A - B)^{-1}g - (A - B)^{-1}(N - N')X. \]

**Lemma 3.1:**

Let \( A \) be nonsingular. Then, (see theorem 2.1 of [9])

\[ \min \left\{ \|\partial A\| : \partial A \text{ is singular} \right\} = \frac{1}{\|A^{-1}\|} \]

Lemma 3.1 explains how a slight perturbation will not change the non-singularity of matrix \( A \).

In the following theorem, some conditions for the solution existence are given.

**Theorem:**

Let \( \tilde{A} = (A, M, N) \) and \( \tilde{B} = (B, M', N') \) be positive fuzzy matrices and \( \tilde{C} = (b, h, g) \) be a positive fuzzy vector. Also, assume \( A - B \) is the product of permutation matrix by diagonal matrix with positive diagonal entries. Moreover, let \( h \geq (M - M')(A - B)^{-1}b \), \( g \geq (N - N')(A - B)^{-1}b \) and \( (M - M')(A - B)^{-1} + I)b \geq h \), then the system \( \tilde{A} \otimes \tilde{X} = \tilde{B} \otimes \tilde{X} + \tilde{C} \) has a positive fuzzy solution.

**Proof:**

Our hypotheses on \( A - B \), implies that \( (A - B)^{-1} \) exists and is a nonnegative matrix (DeMarr 1972). So

\[ X = (A - B)^{-1}b \geq 0. \]

On the other hand

\[ h \geq (M - M')(A - B)^{-1}b \quad \text{and} \quad g \geq (N - N')(A - B)^{-1}b. \]

Thus

\[ Y = (A - B)^{-1}h - (A - B)^{-1}(M - M')X \quad \text{and} \quad Z = (A - B)^{-1}g - (A - B)^{-1}(N - N')X. \]

We have \( Y \geq 0 \) and \( Z \geq 0 \). So \( \tilde{X} = (X, Y, Z) \) is a fuzzy vector which satisfies \( \tilde{A} \otimes \tilde{X} = \tilde{B} \otimes \tilde{X} + \tilde{C} \).

Since \( X - Y = A^{-1}(b - h + (M - M')(A - B)^{-1}b) \), the positivity property of \( \tilde{X} \) can be obtained from \( ((M - M')(A - B)^{-1} + I)b \geq h \).

**Numerical example:**

In this section, we consider two examples of fully fuzzy linear systems and solve them by this numerical procedure.

Example 1. Consider the following DFFLS,
\[
\begin{align*}
\tilde{x}_1 + 6\tilde{x}_2 &= 0.1\tilde{1} + 0.4\tilde{1} + 5\tilde{0} \\
7\tilde{x}_1 + 4\tilde{x}_2 &= 0.1\tilde{1} + 0.4\tilde{1} + 48
\end{align*}
\]

We mean,
\[
\begin{align*}
(5.1, 7, 4) \oplus (x_1, y_1, z_1) \oplus (6.1, 1) \oplus (x_2, y_2, z_2) = (0.1, 0.01, 0.1) \oplus (x_1, y_1, z_1) \oplus (0.1, 0.01, 0.1) \oplus (x_2, y_2, z_2) \oplus (50.10, 17)
\end{align*}
\]

Where
\[
\begin{align*}
A &= \begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix}, \quad M &= \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}, \quad N &= \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}, \quad B &= \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \quad M^r &= \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix} \\
N' &= \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 1 \end{bmatrix}, \quad b &= \begin{bmatrix} 50 \\ 48 \end{bmatrix}, \quad h &= \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad g &= \begin{bmatrix} 17 \end{bmatrix}
\end{align*}
\]

So we get
\[
X = (A - B)^{-1}b = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4.0082 \\ 5.0082 \end{bmatrix},
\]
\[
Y = (A - B)^{-1}h - (A - B)^{-1}(M - M^r)X = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.0317 \\ 0.1528 \end{bmatrix}
\]

and
\[
Z = (A - B)^{-1}g - (A - B)^{-1}(N - N')X = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0.6578 \\ 0.7999 \end{bmatrix}.
\]

Example 2. Consider the following DFFLS,
\[
\begin{align*}
\begin{bmatrix} 4.3, 2 \\ 7, 4 \end{bmatrix}, \quad \begin{bmatrix} 5.2, 1 \\ 10.6, 5 \end{bmatrix}, \quad \begin{bmatrix} 3.0, 3 \\ 2.1, 1 \end{bmatrix}, \quad \begin{bmatrix} 15.5, 4 \\ 15.7, 2 \end{bmatrix}
\end{align*}
\]

So we get
\[
\begin{align*}
x_1 &= 4.3104, \quad y_1 = 0.9102, \quad z_1 = 0.5626 \\
x_2 &= 7.7993, \quad y_2 = 3.7689, \quad z_2 = 7.1764 \\
x_3 &= 4.9776, \quad y_3 = 1.0711, \quad z_3 = 0.5616
\end{align*}
\]

REFERENCES


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