Synchronization Of Chaotic Fractional-Order Chen-Lu Systems Via Active Sliding Mode Control (ASMC)

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Abstract: A chaos synchronization problem means making both chaotic oscillators behave exactly the same. Generally two chaotic systems in synchronization are called a drive system and a response system, respectively. A drive system and a response system may be identical or different. Based on stability theorems in the fractional calculus, analysis of stability is performed for the proposed method. Finally, numerical simulation (synchronizing fractional-order Lu–Chen systems) is presented to show the effectiveness of the proposed controller. The simulations are implemented using two different numerical methods to solve the fractional differential equations.

Key words: Fractional calculus; fractional order active sliding mode controller; synchronization; LU-CHEN.

INTRODUCTION

Although Fractional Order Calculus (FOC) has 300-year of History, its applications in physics and engineering have just begun (R. Hilfer, 2001). In many systems, such as viscoelastic systems (R.L. Bagley, 1991), dielectric polarization, and electromagnetic waves, FOC models exhibited better utility. Furthermore, emergence of effective analytical and numerical methods in differentiation and integration of non-integer (fractional) order equations, in recent years, makes FOC more attractive for the control systems community. Recently, the interest of chaotic synchronization has been extensively grown (G. Chen, 1993; G. Chen, 1998; C.C. Fuh, 1995; X.Y. Wang, 2003; G.R. Wang, 2001). The fact that (Ahmad W., 2003) nonlinear chaotic systems may keep their natural chaotic behavior when their models become fractional has a critical effect in this manner. Chaos synchronization can be applied in many areas such as in chemical reactions, power converters, signal process, communication, and biological systems (Tamasevicius, 1997; Chen, G., 2000; Chen, G., 1998; Lü, J., 2002; Carroll, T.L., 1991). There are many methods for synchronization of a chaotic system such as adaptive control method (C.C. Fuh, 1995; X.Y. Wang, 2003; Yau, H.T., 2004), back stepping control method (Lu, J., 2001), H control method (Slotine, J.E., 1991), sliding mode (variable structure) control method (Chen, C.L., 1997; Yin, X., 2002; Tanaka, K., 1998) and fuzzy control method (Yau, H.T., 2006).

A pioneering work on the concept of “chaotic synchronization” is presented in (L.M. Pecora, 19990). Another's work has been continued through presentation of a method to synchronize two identical chaotic systems with different initial conditions (T.L. Carroll, 1991). Different types of chaotic synchronization methods in terms of complete synchronization, generalized synchronization, phase synchronization and lag synchronization have been reported (M.C. Ho, 2002; G.R. Michael, 1997; E.M. Shahverdiev, 2002; T. Shinbrot, 1993; X.S. Yang, 2001; X. Yu and Y. Song, 2001). Recently, some researchers applied the fractional-order controller to control fractional and integer order dynamics of chaotic systems. In (Hosseinnia, S.H., 2008; Hosseinnia, S.H., 2008) an adaptive fractional controller is proposed to control and synchronize chaos and controller parameter is updated based on a proper adaptation mechanism.

Controller in combination with state feedback is proposed in (S.H. Hosseinnia, 2008). In (S.J. Sadati, 2008) bifurcation in fractional order Newton-Leipnik system was investigated. The projective method is used to synchronize fractional order rigid body system in (M. Shahiri, 1999). Also synchronization of chaotic fractional order Coullet systems has already studied in (M. Shahiri, 2008). In this paper the synchronization of this system will be developed via active sliding mode control with the novelty of synchronizing of two masters and slave systems with different fractional orders. This case attracts the designers, particularly in coding and decoding applications.

Active sliding mode control technique is a discontinuous strategy that relies on two stages of designation. The first step is to select an appropriate active controller to facilitate the design of sliding mode controller. The second stage is to design a sliding mode controller to achieve the synchronization. The rest of the paper is organized as follows.

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This paper is organized as follows. In Section 2, basic definitions in fractional calculus are presented. This section also includes explanation about existing method of approximated solution of fractional differential equations. Based on the active sliding mode control theory, a controller is proposed to synchronize identical and non-identical chaotic fractional-order systems in Section 3. The section comprises of two main parts: design of the controller and analysis of the stability. Numerical simulations results are given in Section 4 to illustrate the effectiveness of the proposed controller. Conclusions in Section 5 close the paper.

II. Fractional-Order Derivative And Its Approximation:

A. Definition:

The differ integral operator, represented by $0^q_t$, is a combined differentiation-integration operator commonly used in fractional calculus and general calculus operator, including fractional-order and integer is defined as:

$$0^q_t = \begin{cases} \frac{dq}{dt} & q > 0 \\ 1 & q = 0 \\ \int_0^t (dt)^{-q} & q < 0 \end{cases}$$

(1)

There are several definitions of fractional derivatives (I. Podlubny, 1999). The best-known one is the Riemann-Liouvile definition, which is given by

$$\frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(q)} \frac{d^n}{dt^n} \int_0^t \frac{f(t')}{(t-t')^{n+1-q}} dt'$$

(2)

Where $n$ is an integer such that $n - 1 < q < n$, $\Gamma(0)$ is the Gamma function. The geometric and physical interpretation of the fractional derivatives was given as follows

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

(3)

The Laplace transform of the Riemann-Liouville fractional derivative is

$$L\left(\frac{d^q f(t)}{dt^q}\right) = s^q L[f(t)] - \sum_{k=0}^{n} s^{q-k} \frac{d^{q-k-1} f(t)}{dt^{q-k-1}}$$

(4)

Where, $L$ means Laplace transform, and $s$ is a complex variable. Upon considering the initial conditions to zero, this formula reduces to

$$L\left(\frac{d^q f(t)}{dt^q}\right) = s^q L[f(t)]$$

(5)

The Caputo fractional derivative of order $q$ of a continuous function $f: R^+ \rightarrow R$ is defined as follows

$$\frac{d^q f(t)}{dt^q} = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t f^{(m)}(t') (t-t')^{m-q-1} dt' & m - 1 < q < m \\ \frac{d^m}{dt^m} f(t) & q = m \end{cases}$$

(6)

Thus, the fractional integral operator of order $\alpha$ can be represented by the transfer function $H(s) = \frac{1}{s^\alpha}$ in the frequency domain.

The standard definition of fractional-order calculus does not allow direct implementation of the fractional operators in time-domain simulations. An efficient method to circumvent this problem is to approximate fractional operators by using standard integer-order operators. In Ref. (A. Charef, 1992), an effective algorithm is developed to approximate fractional-order transfer functions, which has been adopted in (C.G. Li, 2004) and has sufficient accuracy for time-domain implementations. In Table 1 of Ref (W.M. Ahmad, 2003), approximations for $1/s^\alpha$ with $q$ from 0.1 to 0.9 in step 0.1 were given with errors of approximately 2 dB. We will use the $1/s^{0.95}$ approximation formula (C.G. Li, 2004) in the following simulation examples.

$$\frac{1}{s^{0.95}} \approx \frac{1.28318 + 18.6004 + 2.0833}{s}$$

(7)
In the simulation of this paper, we use approximation method to solve the fractional-order differential equations.

III. Designing The Fractional-Order Active Sliding Mode Control And Analysis:

To design the active sliding mode controller, we have procedure a combination of the active controller and the sliding mode controller.

A. Active Sliding Mode Controller Design:

The subsequent objective is to show the efficiency the fractional modeling of dynamic. The method is basically a combination of an active and sliding mode controller. The designation procedure of active sliding mode controller is primarily given and then the stability issue of the proposed method is proven.

Let us, consider a chaotic fractional-order description of the system as follows

\[
0D_{\alpha}^{D}X_1 = A_1X_1 + g_1(X_1) \quad 0 < \alpha < 1 \tag{8}
\]

Where \(X_1(t) \in R^3 \) denotes 3-D state vector, \(A_1 \in R^{3 \times 3} \) represents the linear part of the dynamic and \(g_1: R^3 \rightarrow R^3 \) is the nonlinear part of the system. The procedure description uses asynchronization architecture where equation (8) represents the drive dynamic. Let \(X_0 = (x_{10}, x_{20}, x_{30})^T \) be the initial conditions in the chaos attractor of fractional-orders system (8). Meantime, the response dynamic is defined inclusion of a control signal \(u(t) \in R^3 \) by:

\[
0D_{\alpha}^{D}X_2 = A_2X_2 + g_2(X_2) + u(t) \quad 0 < \alpha < 1 \tag{9}
\]

Where \(X_2(t) \in R^3 \) is the slave 3-D state vector, \(A_2 \) is the same parameter matrices as the drive has, and \(g_2: R^3 \rightarrow R^3 \) simply has the same role as \(g_1 \) in the drive. Synchronization means finding the appropriate control signal \(u(t) \in R^3 \) to derive states of the slave system to evolve as the states of the drive. The synchronization goal will be achieved through the error definition which is as follows:

\[
0D_{\alpha}^{D}X_2 - 0D_{\alpha}^{D}X_1 = A_2X_2 + g_2(X_2) - A_1X_1 - g_1(X_1) + u(t) \tag{10}
\]

Now follow sentence add to the equation (10)

\[
0D_{\alpha}^{D}X_2 - 0D_{\alpha}^{D}X_1 = A_2X_2 + g_2(X_2) - A_1X_1 - g_1(X_1) + u(t) \tag{11}
\]

Thus:

\[
0D_{\alpha}^{D}X_2 - 0D_{\alpha}^{D}X_1 = A_2X_2 + g_2(X_2) - A_1X_1 - g_1(X_1) + u(t) - 0D_{\alpha}^{D}X_2 + 0D_{\alpha}^{D}X_1 \tag{12}
\]

That: \(e = X_2 - X_1 \) and \(A_1 = A_2 = A \)

Thus:

\[
0D_{\alpha}^{D}e = A_2X_2 + g_2(X_2) - A_1X_1 - g_1(X_1) + u(t) - 0D_{\alpha}^{D}X_2 + 0D_{\alpha}^{D}X_1 \tag{12}
\]

Now we assume:

\[
G(X_1, X_2) = g_2(x_2) - g_1(x_1) + (A_2 - A_1)x_1 - 0D_{\alpha}^{D}X_2 + 0D_{\alpha}^{D}X_1 \tag{13}
\]

The aim is to design the controller \(u(t) \in R^3 \) such that:

\[
\lim_{t \to \infty} \|e(t)\| = 0 \tag{14}
\]

Then use with the active control design procedure (D. Matignon, 1996; G. Chen, 1999) \(U(t) \) change as following:

\[
u(t) = H(t) - G(X_1, X_2) \tag{15}
\]

Eq. (15) describes the newly defined control input \(H(t) \).

Where \(H(t) \) is:
H(t) = K w(t) \quad (16)

Where \( k \in R^3 \) is a constant gain vector and \( w(t) \in R \) is the control input that satisfies in:

\[
W(t) = \begin{cases} 
    \frac{w^+(t)}{t} & s(e) \geq 0 \\
    \frac{w^-(t)}{t} & s(e) < 0 
\end{cases}
\]

(17)

Where \( s = s(e) \) is a switching surface that describes the desired dynamics the resultant error is then written by

\[
0^{\alpha_2} e = A e + KW(t) 
\]

(18)

B. Constructing A Sliding Surface:

Constructing a sliding surface which represents a desired system dynamics and the sliding surface described as follows

\[
s(e) = Ce \quad (19)
\]

Where \( C \in R^3 \) is a constant vector. An equivalent control is found when \( \dot{S}(e) = 0 \) which is a necessary condition for the state trajectory to stay on the switching surface \( S(e) = 0 \) hence, the controlled system satisfies the following conditions in the steady state:

\[
S(e) = 0 \quad \text{and} \quad \dot{S}(e) = 0 
\]

(20)

Based on equation (17) to (19), it could be deduced:

\[
\dot{S}(e) = \left( C0^{1-q_2} (e) \right) = 0 
\]

(21)

Thus,

\[
0^{1-q_2} w(t) = -(ck)^{-1}CA \left( 0^{1-q_2} e(t) \right) 
\]

(22)

A solution of Eq.21 is

\[
w_{eq}(t) = -(CK)^{-1}CAe(t) 
\]

(23)

C. Sliding Mode Control of Fractional Order System:

We consider the constant plus proportional rate reaching law will be considered (H. Zhang, 2004). Accordingly the reaching law is obtained as:

\[
0^{\alpha_2} S = -\rho \ sgn \ (s) - rs 
\]

(24)

That \( sgn(0) \) represents the sign function. They \( \rho, r \) are gains that the sliding conditions Eq. (19) are satisfied. From Esq. (17), (18) have:

\[
0^{\alpha_2} S = C0^{\alpha_2} e = C[Ae + kw(t)] 
\]

(25)

From Esq. (23) and (24) find control effort can be defined as:

\[
w(t) = -(CK)^{-1}[C(\alpha_1 + A)e + \rho sgn(s)]. 
\]

(26)

D. Stability:

First, we represent stability theorems from the fractional calculus.

**Theorem 1** (Matignon (D. Matignon, 1996)). The following system:

\[
0^{\alpha_i} x = Ax, \quad x(0) = x_0 
\]

(27)
Where $0 < q < 1, x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is asymptotically stable if $|\text{arg}(\text{eig}(A))| > \frac{q\pi}{2}$.

According to Theorem 1, as long as all eigenvalues of $[A - K(CK)C(rI + A)]$ ($\lambda_i = 1,2,3$) satisfy the conditions $|\text{arg}(\lambda_i)| > \frac{q\pi}{2}$, the system is asymptotically stable.

**Fig. 1:** Stability region of linear fractional-order system with $q$.

**IV. Numerical Simulations:**

**A. Synchronization Between Two Fractional-Order Lu-Chen Systems:**

The LU system (G. Chen, 1999), was introduced by Chen and Lu

$$
\begin{align*}
0D^\alpha_x x &= \rho(y - x) \\
0D^\alpha_y y &= -xz + vy \\
0D^\alpha_z z &= xy - \mu z
\end{align*}
$$

(28)

For this system matrix $A_1$ is

$$
A_1 = \begin{bmatrix}
-\rho & \rho & 0 \\
0 & \nu & 0 \\
0 & 0 & -\mu
\end{bmatrix}
$$

(29)

And the Chen system was introduced by Chen and Ueta in 1999.

$$
\begin{align*}
0D^\alpha_x x &= a(y - x) \\
0D^\alpha_y y &= (c - a)x - xz + cy \\
0D^\alpha_z z &= xy - bz
\end{align*}
$$

(30)

For this system matrix $A_2$ is

$$
A_2 = \begin{bmatrix}
-a & a & 0 \\
c - a & c & 0 \\
0 & 0 & -b
\end{bmatrix}
$$

(31)

In this section, we consider using (ASMC) technique to obtain synchronization. This controller guarantees the synchronization two fractional orders Lu-Chen systems with the following initial conditions:

$(x_{10}, y_{10}, z_{10}) = (1,1,1)$

and

$(x_{20}, y_{20}, z_{20}) = (3, -6,9)$.

Consider two fractional orders LU–Chen systems as drive and response systems respectively:

$$
\begin{align*}
0D^\alpha_{t} x_1 &= 35(y_1 - x_1) \\
0D^\alpha_{t} y_1 &= -x_1z_1 + 28y_1 \\
0D^\alpha_{t} z_1 &= x_1y_1 - 3z_1
\end{align*}
$$

(32)
Response system.

\[
\begin{align*}
0^{Dx_2} x_1 &= 35(y_2 - x_2) \\
0^{Dy_2} y_2 &= -7x_2 - x_2z_2 + 28y_2 \\
0^{Dz_2} z_2 &= x_2y_2 - 3z_2
\end{align*}
\]  \hspace{1cm} (33)

Assume that order of the drive is \(q_1 = 0.88\) and order the response is \(q_2=0.9\). Parameters of the controller are chosen as \(k = [-2, -9, -2]^T, C = [1, 1, -1], r = 75\) and \(\rho = 0.35\). This selection of parameters results in eigenvalues \(\lambda_1, \lambda_2, \lambda_3 = [-75, -58.33, -6]\) which located in a stable region \(\arg(\lambda_i) > \frac{q_1\pi}{2}\).

Fig. shows the effectiveness of the proposed controller to synchronize two fractional-order modeled systems. It should be noted that control \(u(t)\), has been activated at \(t = 0\), the simulation results are shown in Fig.2.
Fig. 2: Results of simulation.

V. Conclusion:
In this paper, synchronization of two chaotic Lu-Chen system with fractional orders models (as drive and response) with different orders is investigated. Active sliding mode method has been developed to imply the task. It has been shown that by proper selection of the control parameters \( r, k \) and \( c \), the drive and response systems are synchronized. Furthermore, all Eigen values of synchronized system will satisfy the suffusion condition i.e. \( |\arg(e^{i\theta})A| > \pi/2 \). This means the error is stabilized so in the long term analysis, the synchronization would be guaranteed.

The proposed method has been developed to synchronize two drive and response system with different fractional-order. Numerical simulations have shown the efficiency of the proposed controller in the mentioned task.
REFERENCES


