On The Central Value of Fuzzy Numbers

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Abstract: Ranking fuzzy numbers plays a very important role in linguistic decision making and some other fuzzy application systems. Many methods have been proposed to deal with ranking fuzzy numbers. This paper, proposed a central value method to rank fuzzy numbers. The method can effectively rank various fuzzy numbers and their images. The researchers also used some comparative examples to illustrate the advantage of the proposed method.

Key words: Ranking; Fuzzy numbers; Median; Central value; Defuzzification.

INTRODUCTION

In most real situation, one is forced to take decision on the basis of ill-defined variables and imprecise data. The theory of fuzzy sets is a natural tool to model this situation as fuzzy numbers well represent imprecise quantities. In decision-making problems we have the necessity to optimize some procedure and so it is necessary to have ranking of the quantities involved. Many authors have studied different definitions of ranking on the set of fuzzy numbers F, (Bortolan and Degani (1985)). Most of these are based on the definition of an evaluation function (F-evaluation function), which maps fuzzy numbers in the real line. The order on F is induced by the real number total order. As fuzzy numbers are intervals in which boundaries are blurred, the difficulty in ranking them arise from the problem created in ranking real intervals. When the supports of fuzzy numbers are disjointed, there are no problems and all the methods lead to the same solution. But the decision is not evident when the intersection between supports is not empty. In these cases, it seems that the solution depends on subjective elements depending of the nature of the problem and the decision-maker. Having reviewed the previous methods, this article proposes a method to use the concept of Central value, so as to find the order of fuzzy numbers. This method can distinguish the alternatives clearly. The main purpose of this article is that, the central value can be used as a crisp approximation of a fuzzy number. Therefore, by the means of this defuzzification, this article aims to present a new method for ranking of fuzzy numbers. In addition to its ranking features, this method removes the ambiguities resulted from the comparison of previous ranking. In Section 2, we recall some fundamental results on fuzzy numbers. In Section 3, a crisp approximation of a fuzzy number is obtained. In this Section some remarks and proposed method for ranking fuzzy numbers are illustrated.

2. Basic Definition and Notation:

The basic definitions of a fuzzy number are given in (Kauffman et al., 1991; Zimmerman, 1991; Zadeh, 1965; Saneifard et al. 2009a; Saneifard, 2009c; Saneifard, 2007) as follows:

Definition 2.1:
A fuzzy number $A$ is a mapping $\mu_A(x) : \mathbb{R} \rightarrow [0,1]$ with the following properties:
1. $\mu_A$ is an upper semi-continuous function on $\mathbb{R}$,
2. $\mu_A(x) = 0$ outside of some interval $[a_1, b_2] \subset \mathbb{R}$.
3. There are real numbers $a_2, b_1$ such that $a_1 \leq a_2 \leq b_1 \leq b_2$ and
   3.1 $\mu_A(x)$ is a monotonic increasing function on $[a_1, a_2]$,
   3.2 $\mu_A(x)$ is a monotonic decreasing function on $[b_1, b_2]$,
   3.3 $\mu_A(x) = 1$ for all $x$ in $[a_2, b_1]$.

The set of all fuzzy numbers is denoted by $F$.

Let $\mathbb{R}$ be the set of all real numbers. We assume a fuzzy number $A$ that can be expressed for all $x \in \mathbb{R}$ in the form
where \( a, b, c \) and \( d \) are real numbers such that \( a < b \leq c < d \) and \( g \) is a real valued function that is increasing and right continuous and \( h \) is a real valued function that is decreasing and left continuous. Notice that (2.1) is an LR fuzzy number. A fuzzy number \( A \) with shape function \( g \) and \( h \) defined by

\[
g(x) = \left(\frac{x-a}{b-a}\right)^n, \tag{2.2}\]

and

\[
h(x) = \left(\frac{d-x}{d-c}\right)^n, \tag{2.3}\]

respectively, where \( n > 0 \), will be denoted by \( A = \langle a, b, c, d \rangle_n \). If \( n = 1 \), we simply write \( A = \langle a, b, c, d \rangle \), which is known as a trapezoidal fuzzy number. Each fuzzy number \( A \) described by (2.1) has the following \( \alpha \)-level sets (\( \alpha \)-cut) \( A_\alpha = \{a_\alpha, b_\alpha\}, a_\alpha, b_\alpha \in \mathbb{R}, \alpha \in [0,1] \)

1. \( A_\alpha = [g^{-1}(\alpha), h^{-1}(\alpha)] \) for all \( \alpha \in [0,1] \),
2. \( A_0 = [b, c] \),
3. \( A_1 = [a, d] \).

If \( A = \langle a, b, c, d \rangle_n \) then for all \( \alpha \in [0,1] \),

\[
A_\alpha = [a + \alpha^n (b-a), d - \alpha^n (d-c)], \tag{2.4}
\]

Another important notion connected with fuzzy number \( A \) is an cardinality of a fuzzy number \( A \).

**Definition 2.2:** (Bodjanova, 2005). Cardinality of a fuzzy number \( A \) described by (2.1) is the value of the integral

\[
\text{card} \ A = \int_a^b A(x)dx = \int_0^1 (b_\alpha - a_\alpha) d\alpha. \tag{2.5}
\]

If \( A = \langle a, b, c, d \rangle_n \) then

\[
\text{card} \ A = \frac{b-a}{n+1} + (c-d) + \frac{d-c}{n+1} \tag{2.6}
\]

In this paper we will always refer to fuzzy number \( A \) described by (2.1).

**Definition 2.3:** (Bodjanova, 2005). The median value of a fuzzy number \( A \) characterized by (2.1) is the real number \( m_A \) from the support of \( A \) such that

\[
\int_a^{m_A} A(x)dx = \int_{m_A}^d A(x)dx, \tag{2.7}
\]

For practical purposes expression (2.7) can be rewritten as...
\[ \int_{a}^{m_A} A(x) dx = 0.5 \text{card } A. \quad (2.8) \]

The article can classify fuzzy numbers with respect to the “distribution” of their cardinality as follows: a fuzzy number \( A \) is called

1. A fuzzy number with equally heavy tails if
\[ \int_{a}^{b} A(x) dx = \int_{c}^{d} A(x) dx, \]

2. A fuzzy number with light tails if
\[ \max \left\{ \int_{a}^{b} A(x) dx, \int_{c}^{d} A(x) dx \right\} \leq 0.5 \int_{a}^{d} A(x) dx, \]

3. A fuzzy number with heavy left tail if
\[ \int_{a}^{b} A(x) dx > 0.5 \int_{a}^{d} A(x) dx, \]

4. A fuzzy number with heavy right tail if
\[ \int_{c}^{d} A(x) dx > 0.5 \int_{a}^{d} A(x) dx. \]

Now the article will study location of the median value \( m_A \) in the support of \( A \). The article will also identify the fuzziness of \( m_A \) determined by its membership grade \( A(m_A) \).

**Proposition 2.1:**
(Bodjanova, 2005). If \( A \) is a fuzzy number with light tails then
\[ m_A = \frac{b + c}{2} + 0.5 \left( \int_{c}^{d} A(x) dx - \int_{a}^{b} A(x) dx \right). \quad (2.9) \]

and \( A(m_A) = 1 \).

**Corollary 2.1:**
If \( A \) has equally heavy tails then \( m_A = \frac{b + c}{2} \) and \( A(m_A) = 1 \). Obviously, if \( A \) has a heavy left (right) tail then \( m_A \) is not located in the core of \( A \) but it is shifted to the left (right) end of the support of \( A \). For a trapezoidal fuzzy number \( A \) and for its modifications by selected linguistic hedges the article will provide formulas for the location of the median value.

**Proposition 2.2:**
(Bodjanova, 2005). Let \( A = \langle a, b, c, d \rangle_a \). Then
\[ m_A = a + \left( \frac{(b - a)^n}{2} - (n + 1) \text{card } A \right)^{\frac{1}{n+1}}, \quad (2.10) \]

if \( A \) has a heavy left tail, and
\[ m_A = d - \left( \frac{(d - c)^n}{2} - (n + 1) \text{card } A \right)^{\frac{1}{n+1}}, \quad (2.11) \]

if \( A \) has a heavy right tail.
Corollary 2.2:
Let $\mathbf{A} = \langle a, b, c, d \rangle$ be a trapezoidal fuzzy number. Then

$$m_A = a + \sqrt{(b-a) \text{card } A},$$

if $A$ has a heavy left tail, and

$$m_A = d - \sqrt{(d-c) \text{card } A},$$

if $A$ has a heavy right tail.

When the median value is located outside the core of a fuzzy number $A$, it belongs to $A$ with the membership grade less than 1 (Bodjanova, 2005). The median value $m_A$ can be used as a scalar representative of the center of a fuzzy number $A$. There are two other representatives of the center of $A$ described in the literature (Dubios et al., 2000). The center of gravity of the support of $A$ weighted by the membership grad

$$g_A = \frac{\int_a^d x \mu_A(x) dx}{\int_a^d \mu_A(x) dx},$$

and the center of the core (the central modal value) of $A$

$$m_{oA} = \frac{b+c}{2}.$$  (2.15)

The value of $m_{oA}$ does not take into account the shape of the membership function of $A$. Fuzzy numbers with the same core have the same center of core regardless of their tails. Therefore $m_{oA}$ represents only the crisp part of $A$. The value of $g_A$ takes into account the entire membership function of $A$. If two fuzzy numbers $A$ and $B$ have the same core and the same support then, in general, $g_A \neq g_B$. If $A$ has a heavy right (left) tail, then $g_A$ tends to move towards the right (left) end of the support of $A$. The heavier is the tail the further is the value of $g_A$ from the value of $m_{oA}$ and the membership grade $A(g_A)$ becomes smaller. The median value $m_A$ is a compromise between $m_{oA}$ and $g_A$. It takes into account the membership function of $A$ and, because in the case of a heavy tail it is located between $g_A$ and $m_{oA}$, the membership grade of $m_A$ in $A$ is larger than that of $g_A$. Therefore, the researchers recommend the median value as a scalar representative of a fuzzy number with a heavy tail. This article used a scalar representative of a fuzzy number based on the weighted average of the crisp representative of the fuzzy center of fuzzy number described above.

3. Comparison of Fuzzy Numbers Using A Central Value:
In this section, the researchers will propose the ranking of fuzzy numbers associated with the central value.

Definition 3.1:
(Bodjanova, 2005). Let $A$ be an arbitrary fuzzy number and $g_A$, $m_{oA}$ and $m_A$ be the center of gravity, the center of core and the median value of $A$, respectively. Then the central value of $A$ is given by

$$c_A = \frac{g_A A(g_A) + m_{oA} A(m_{oA}) + m_A A(m_A)}{A(g_A) + A(m_{oA}) + A(m_A)}.$$  (3.16)

The central value can be used as a crisp approximation of a fuzzy number, so, the resulting scalar value is used to rank the fuzzy numbers.
Remark 3.1:
If \( \inf Supp(A) \geq 0 \) or \( \inf a_a \geq 0 \) then \( c_A \geq 0 \).

Remark 3.2:
If \( \sup Supp(A) < 0 \) or \( \sup b_a < 0 \) then \( c_A < 0 \).

Remark 3.3:
For any two symmetric trapezoidal fuzzy numbers \( A = \langle a_0, b_0, b_1, d_1 \rangle \) and \( B = \langle a_1, b_0, b_0, d_1 \rangle \), \( c_A = c_B \).

Remark 3.4:
For two arbitrary fuzzy numbers \( A \) and \( B \), we have \( c_{A+B} = c_A + c_B \).

Remark 3.5:
For all symmetric trapezoidal fuzzy numbers \( A = \langle -a, b, b, a \rangle \), \( c_A = 0 \).

In the following, we present a new approach for ranking fuzzy numbers based on the distance method. The method not only considers the central value of a fuzzy number, but also considers the maximum crisp value of fuzzy numbers. For ranking fuzzy numbers, this study firstly defines a maximum crisp value \( \tau_{\max} \) to be the benchmark and its characteristic function \( \mu_{\tau_{\max}}(x) \) is as follows:

\[
\mu_{\tau_{\max}}(x) = \begin{cases} 
1, & \text{when } x = \tau_{\max}, \\
0, & \text{when } x \neq \tau_{\max}. 
\end{cases}
\]  
(3.17)

When ranking \( n \) fuzzy numbers \( A_1, A_2, \ldots, A_n \), the maximum crisp value \( \tau_{\max} \) is defined as:

\[
\tau_{\max} = \max \{ x | x \in \text{Domain}(A_1, \ldots, A_n) \}. 
\]  
(3.18)

Assume that there are \( n \) fuzzy numbers \( A_1, A_2, \ldots, A_n \). The proposed method for ranking fuzzy numbers \( A_1, A_2, \ldots, A_n \) is now presented as follows:

Use the point \( (c_{A_j}, 0) \) to calculate the ranking value \( CV_{A_j} = D(c_{A_j}, \tau_{\max}) \) of the fuzzy numbers \( A_j \), where \( 1 \leq j \leq n \), as follows:

\[
D(c_{A_j}, \tau_{\max}) = |c_{A_j} - \tau_{\max}|. 
\]  
(3.19)

From formula (3.19), we can see that \( CV_{A_j} = D(c_{A_j}, \tau_{\max}) \) can be considered as the Euclidean distance between the point \( (c_{A_j}, 0) \) and the point \( (\tau_{\max}, 0) \). We can see that the smaller the value of \( CV_{A_j} \), the better the ranking of \( A_j \), where \( 1 \leq j \leq n \).

Definition 3.2:
Let, there are \( n \) fuzzy numbers \( A_1, A_2, \ldots, A_n \), and \( CV_{A_j} \) are the central value of their. Define the ranking of \( A_j \) by \( CV(\cdot) \) on \( F \), i.e.

1. \( CV_{A_j} > CV_{A_j} \) if only if \( A_j > A_j \),
2. \( CV_{A_j} < CV_{A_j} \) if only if \( A_j > A_j \),
3. \( CV_{A_j} = CV_{A_j} \) if only if \( A_j = A_j \).

Then, this article formulates the order \( \geq \) and \( \leq \) as \( A_j \geq A_j \) if and only if \( A_j > A_j \) or \( A_j = A_j \), \( A_j \leq A_j \) if and only if \( A_j < A_j \) or \( A_j = A_j \).

3.1 Using The Proposed Ranking Method In Fuzzy Multi-Criteria Decision Making Based on An FN-IOWA Operator:
Chen and Chen (Chen et al., 2003) proposed a method to handle fuzzy multi-criteria decision making problems based on fuzzy number induced ordered weighted averaging (FN-IOWA) operator and applied the
algorithm to a human selection problem. In this section, we use the same example illustrated in Chen and Chen to show the efficiency of the proposed ranking method. For more detailed information about the FN-IOWA operator, (see R. R. Yager 1998, R. R. Yager and D. P. Filev 1999, Chen and Chen 2003). Here we just pay attention to the fuzzy ranking step in the final decision making process. A new manager will be recruited among three candidates, X, Y and Z. The final scores, which can be obtained by an FN-IOWA operator, are fuzzy numbers and are listed as follows:

\[
S_X = (0.2501, 0.7727, 2.2501), \\
S_Y = (0.0667, 0.5000, 1.8750), \\
S_Y = (0.1667, 0.6592, 2.2500). \\
\]

By applying the proposed ranking method, the index central value of each alternative can be obtained as follows:

\[
CV_X = 0.96, \\
CV_Y = 1.21, \\
CV_Z = 1.16. \\
\]

We can see that their ranking order X > Z > Y. Therefore, Candidate X is more suitable than Candidate Z, and Candidate Z is more suitable than Candidate Y. The result are the same as the one presented in Chen and Chen.

4. Conclusion:
In this paper, the researchers proposed the central value method to rank fuzzy numbers. This method use a crisp set approximation of a fuzzy number. The method can effectively rank various fuzzy numbers and their images (normal/non-normal/trapezoidal/general). The calculations of this method are simpler than the other approaches.

REFERENCES


