Legendre Wavelet Method For Solving Differential Algebraic Equations

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Abstract: In this paper the aim is to solve the differential-algebraic equations using Legendre wavelet method. The advantage of using of the proposed method are considered. Two illustrations are given to show the high accuracy and efficiency of the proposed method particularly for nonlinear functional differential equations.

Key words: Legendre wavelet method; Differential algebraic equations; Hessenberg forms of differential algebraic equations.

INTRODUCTION

Differential algebraic equations (DAEs) play a dominant role in many branches of science and engineering. DAEs have received considerable interest in recent years and have been extensively investigated and applied for many real problems which are modeled in various areas. For instances, circuit analysis, computer-aided design, power systems, simulation of mechanical systems and more general optimal control problems; thus, they have attracted the attention of numerical analysts (Brenan et al., 1989: Ascher and Petzold 1998).

The index of a DAE is a measure of the degree of singularity of the system, and it is widely regarded as an indication of certain difficulties for numerical solution of ODE systems (Brenan et al., 1989) The following statements and discussion of DAEs have been described by Celik (Celik 2004) The most regular form of a DAE is as follows:

\[ F(t, y, y') = 0, \]

where \( \frac{\partial F}{\partial y} \) may be singular. The rank and structure of this Jacobian matrix depends, in general, on the solution \( y(t) \), and for simplicity, we shall always assume that it is independent of \( t \). The important case of a semi-explicit DAE, or an ODE with constraints, is given in the following equations:

\[ \begin{align*}
    x' &= f(t, x, z), \\
    0 &= g(t, x, z).
\end{align*} \]

This is a special case of equation (1). The index is 1 if \( \frac{\partial g}{\partial z} \) is non-singular, because then one differentiation of equation (2) yields \( z' \) in principle. For the semi-explicit index-1 DAE, we can differentiate between differential variables \( x(t) \) and algebraic variables \( z(t) \). The algebraic variables may be less smooth than the differential variables by one derivative. In the general case, each component of \( y' \) may contain a combination of differential and algebraic components, which make the numerical solution of such high-index problems much more difficult. The semi-explicit form is de-coupled in this sense. On the other hand, any DAE can be written in the semi-explicit form (2), but with the index increased by 1 upon defining \( y' = z \), which gives the following equations:

\[ \begin{align*}
    y' &= z, \\
    0 &= F(t, y, z).
\end{align*} \]

It is evident that re-writing by itself does not make the problem easier to solve. The converse transformation is also possible. Given a semi-explicit index-2 DAE system with \( z = w' \), the system is easily represented as follows:

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\[ x' = f(t, x, w'), \]
\[ 0 = g(t, x, w'), \]  

(4)

where this system is an index-1 DAE and yields exactly the same solution for \( x \) as (2) above. The classes of fully-implicit index-1 DAEs of the form (1) and semi-explicit index-2 DAEs of the form (2) are therefore equivalent. We define the index of a DAE. For general DAE systems (1), the index along a solution \( y(t) \) is the minimum number of differentiations of \( y \) and \( t \) that the system requires to solve for \( y' \) uniquely in terms of \( y \) and \( t \) (i.e., to define an ODE for \( y \)). Thus, the index is defined in terms of the over-determined system as follows:

\[ \frac{\partial^n F}{\partial t^n}(t, y, y', \ldots, y^{(m-1)}) = 0, \]  

(5)

where \( m \) is assumed to be the smallest integer such that \( y' \) in (5) can be determined in terms of \( y \) and \( t \).

It should be noted that in practice, differentiating the system as in (5) is rarely done in a computation. However, such a definition is useful in understanding the underlying mathematical structure of the DAE system and therefore, in selecting an appropriate numerical method (Ascher and Petzold 1998: Celik 2004).

In recent years, considerable efforts have been made to solve systems of DAEs. There are some numerical methods that solve DAEs; the important ones are the Pade approximate series (Celik 2004) collocation method the method which use both BDF (Brenan et al., 1989: Gear and Petzold, 1984: Ascher and L.R. Petzold 1998: Ascher and Spiter 1994) implicit Runge-Kutta methods (Brenan et al., 1989: Ascher and R.J. Spiter 1994: Ascher and L.R. Petzold 1991). Multiquadric approximation scheme (Karimi Vanani et al., 2011) homotopy perturbation method (Soltanian 2010) and Adomian decomposition method (Hosseini 2005) In this paper, we are interested in solving DAEs using LWM as a fast method with an easy resolvent algorithm.

The organization of this paper is as follows. Section 2 gives some notations and basic definitions of the DAE forms. In Section 3, basic concepts of the Legendre wavelet method (LWM) are given. The application of the LWM on DAEs is presented in Section 4. To show the efficiency of the method, two illustrative numerical experiments are presented in Section 5. Finally, Section 6 consists of some obtained conclusions.

2. Special differential algebraic equation forms:

Many practical problems with higher-indices that arise in applied sciences can be considered to be a combination of more restrictive structures of ODEs coupled with constraints. In such systems, the algebraic and differential variables are explicitly identified for higher-index DAEs as well, and the algebraic variables may all be eliminated using the same number of differentiations. These are called Hessenberg forms of the DAE and are given as follows (Brenan 1989).

2.1. Hessenberg index-1:

\[ x' = f(t, x, z), \]
\[ 0 = g(t, x, z). \]  

(6)

In the above equations, the Jacobian matrix function \( \frac{\partial g}{\partial z} \) is assumed to be non-singular for all \( t \). This system is also often referred to as a semi-explicit index-1 system. Semi-explicit index-1 DAEs are closely related to implicit ODEs. Using the implicit function theorem ([12, pp. 36-37]), we can actually solve for variable \( z \) in constraint (6). Substituting the result in differential equation (6) yields an ODE in variable \( x \).

2.2. Hessenberg index-2:

\[ x' = f(t, x, z), \]
\[ 0 = g(t, x), \]  

(7)
where the product of Jacobians \( \frac{\partial g}{\partial x} \frac{\partial f}{\partial x} \) is non-singular for all \( t \) s. Note the absence of the algebraic variable \( z \) from constraint (7). This system is a pure index-2 DAE, and all algebraic variables play the role of index-2 variables.

2.3. Hessenberg index-3:

\[
x' = f(t,x,y,z), \\
y' = g(t,x,y), \\
0 = h(t,y),
\]

(8)

where the product of three matrix functions \( \frac{\partial h}{\partial y}, \frac{\partial g}{\partial y} \) is non-singular. The index of a Hessenberg DAE is found, as in the general case, by differentiation. However, only algebraic constraints must be differentiated (Ascher and Petzold 1991).

3. Legendre wavelet method:

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously, we have the following family of continuous wavelets as (Gu, W.S. Jiang, 1996).

\[
\psi_{a,b}(t) = a^{-1/2} \psi \left( \frac{t - b}{a} \right), \quad a, b \in \mathbb{R}, a \neq 0,
\]

where \( a \) is the dilation parameter and \( b \) is the translation parameter. If we restrict the parameters \( a \) and \( b \) to discrete values as \( a = a_0^{-k}, b = nb_0^{-k}, a_0 > 1, b_0 > 0 \) and \( n, k \in \mathbb{N} \), we have the following family of discrete wavelets,

\[
\psi_{k,n}(t) = a_0^{-k/2} \psi \left( a_0^k t - nb_0 \right),
\]

where \( \psi_{k,n}(t) \) form a wavelet basis for \( L^2(\mathbb{R}) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \) then \( \psi_{k,n}(t) \) forms an orthonormal basis (Razzaghi and Yousefi 2000) Legendre wavelet \( \psi_{a,m}(t) = \psi(k,\hat{n},m,t) \) has four arguments; \( \hat{n} = 2n - 1, n = 1, 2, 3, ..., 2^{k-1}, k \) can assume any positive integer, \( m \) is the order for Legendre polynomials and \( t \) is the normalized time. They are defined on the interval \([0,1)\) as (Razzaghi and Yousefi 2000).

\[
\psi_{a,m}(t) = \begin{cases} 
2 \sqrt{m+1} \frac{2}{2^k} P^2 \left( 2^k \frac{t - \hat{n}}{2^k} \right), & \hat{n} - 1 \leq t < \hat{n} + \frac{1}{2^k}, \\
0, & \text{otherwise},
\end{cases}
\]

where \( m = 0, 1, ..., M - 1, n = 1, 2, 3, ..., 2^{k-1} \). The coefficient \( \sqrt{m+1} \) is for orthonormality, the dilation parameter is \( a = 2^{-k} \) and translation parameter is \( b = \hat{n}2^{-k} \). Here, \( P_n(t) \) are the well-known Legendre polynomials of order \( m \) which are defined on the interval \([-1,1]\), and can be determined with the aid of the following recurrence formulae,

\[
P_0(t) = 1, \quad P_1(t) = t, \\
P_{m+1}(t) = \left( \frac{2m+1}{m+1} \right) t P_m(t) - \left( \frac{m}{m+1} \right) P_{m-1}(t), m = 1, 2, \ldots
\]

A function \( f(t) \) defined over \([0,1)\) may be expanded as
\[ f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{n,m}(t), \]  

(9)

where \( c_{nm} = \langle f(t), \psi_{n,m}(t) \rangle \), in which \( \langle , , \rangle \) denotes the inner product. If the infinite series in equation (1) is truncated, then it can be written as

\[ y(t); \sum_{n=1}^{2^{k-1}M-1} \sum_{m=0}^{\infty} c_{nm} \psi_{n,m}(t) = C^T \Psi(t), \]  

(10)

where \( C \) and \( \Psi(t) \) are \( 2^{k-1}M \times 1 \) matrices given by

\[ C = [c_{1,0}, \ldots, c_{1,M-1}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}]^T, \]  

(11)

\[ \Psi(t) = [\psi_{1,0}(t), \ldots, \psi_{1,M-1}(t), \psi_{2,0}(t), \ldots, \psi_{2,M-1}(t), \ldots, \psi_{2^{k-1},0}(t), \ldots, \psi_{2^{k-1},M-1}(t)]^T. \]  

(12)

4. Numerical solution of DAEs:

In this section we are interested in solving DAEs. Let us consider the following Hessenberg index-3 on \([0,1]\),

\[
\begin{align*}
x' &= f(t, x, y, z), \\
y' &= g(t, x, y), \\
0 &= h(t, y)
\end{align*}
\]  

(13)

As mentioned in previous section, the approximate solution for the function \( u \), is as:

\[ u(t) = C^T \Psi(t), \]

thus for the solution of above DAE it is sufficient to suppose that

\[ x(t) = C^T \Psi_1(t), \quad y(t) = C^T \Psi_2(t). \]

Therefore, we have:

\[
\begin{align*}
C^T \Psi_1' &= f(t, C^T \Psi_1, C^T \Psi_2, C^T \Psi_2), \\
C^T \Psi_2' &= g(t, C^T \Psi_1, C^T \Psi_2), \\
0 &= h(t, C^T \Psi_2).
\end{align*}
\]

(14)

Now, in order to apply LWM we are required to \( 2^{k-1}M \) collocation points. We have selected them in \([0,1]\), with equal spaces. Thus, implementing the collocation points, we obtain:

\[
\begin{align*}
C^T \Psi_1'(t_i) &= f(t_i, C^T \Psi_1(t_i), C^T \Psi_2(t_i), C^T \Psi_2(t_i)), \\
C^T \Psi_2'(t_i) &= g(t_i, C^T \Psi_1(t_i), C^T \Psi_2(t_i)), \\
0 &= h(t_i, C^T \Psi_2(t_i)).
\end{align*}
\]

The above DAE yields \( 2^{k}M - 2 \) equations, and initial condition produce two equations. Thus the system has \( 2^{k}M \) equations and \( 2^{k}M \) unknowns. Therefore after solving the system, the unknown coefficients will be found.

Similarly this process can be applied for other DAEs systems, easily.
5. Numerical examples:
In this section, two experiments of DAEs are given and some comparisons are made to illustrate the efficiency of the method. The computations associated with the experiments discussed above were performed in Maple 14 on a PC with a CPU of 2.4 GHz.

Problem 5.1.
Consider following DAE of index-2,
\[
\begin{align*}
\dot{x}_1(t) &= \lambda z(t) + p_1(t), \quad t \in [0,1] \\
\dot{x}_2(t) &= (\lambda - 5)z(t) + p_2(t), \\
0 &= t^2 x_1(t) + \sin(t) x_2(t) + s(t), \\
x_1(0) &= 1, x_2(0) = 1, z(0) = 2,
\end{align*}
\]
where
\[
\begin{align*}
p_1(t) &= e^t - 10(e^t + e^{-t}), \\
p_2(t) &= -(6e^{-t} + 5e^t), \\
s(t) &= -(t^2 e^t + \sin(t)e^{-t}),
\end{align*}
\]
The exact solution is
\[
\begin{align*}
x_1(t) &= e^t, \\
x_2(t) &= e^{-t}, \\
z(t) &= e^{-t} + e^t,
\end{align*}
\]
We have solved this experiment using LWM with different \( M \) and \( k = 1 \). The obtained results including the maximum absolute error for \( x_1(t), x_2(t), y(t) \) and \( \lambda = 15 \) are given in Table 1.

<table>
<thead>
<tr>
<th>( M )</th>
<th>Max.err of ( x_1(t) )</th>
<th>Max.err of ( x_2(t) )</th>
<th>Max.err of ( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>( 2.29 \times 10^{-5} )</td>
<td>( 1.58 \times 10^{-5} )</td>
<td>( 1.16 \times 10^{-6} )</td>
</tr>
<tr>
<td>7</td>
<td>( 2.14 \times 10^{-6} )</td>
<td>( 1.39 \times 10^{-6} )</td>
<td>( 1.14 \times 10^{-7} )</td>
</tr>
<tr>
<td>8</td>
<td>( 5.80 \times 10^{-8} )</td>
<td>( 3.97 \times 10^{-8} )</td>
<td>( 3.33 \times 10^{-9} )</td>
</tr>
<tr>
<td>9</td>
<td>( 4.18 \times 10^{-9} )</td>
<td>( 2.75 \times 10^{-9} )</td>
<td>( 2.37 \times 10^{-10} )</td>
</tr>
<tr>
<td>10</td>
<td>( 9.13 \times 10^{-11} )</td>
<td>( 6.23 \times 10^{-11} )</td>
<td>( 5.55 \times 10^{-12} )</td>
</tr>
<tr>
<td>11</td>
<td>( 5.23 \times 10^{-12} )</td>
<td>( 3.47 \times 10^{-12} )</td>
<td>( 3.07 \times 10^{-13} )</td>
</tr>
<tr>
<td>12</td>
<td>( 7.4 \times 10^{-14} )</td>
<td>( 4.84 \times 10^{-14} )</td>
<td>( 5.0 \times 10^{-15} )</td>
</tr>
</tbody>
</table>

From the numerical results in Table 1, it is easy to conclude that obtained results by the proposed algorithm are in good agreement with the exact solution.

Problem 5.2.
Consider following Hessenberg DAE of index-3,
\[
A(t)\dot{x}(t) + B(t)x(t) = 0
\]
with
\[
A(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} t & -t^3 & 1 & t \\
0 & t & 0 & -3 \\
- (1 - t^2) & -t^3 & 0 & t^2 \\
t & t & -1 & -1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\
1 \\
0 \\
0 \end{bmatrix}
\]
The exact solution is
\[
x(t) = \begin{bmatrix} e^{t^2}, e^{t^2}, te^{-t^2}, te^{t^2} \end{bmatrix}^T.
\]
We have solved this experiment using LWM with different $M$ and $k = 1$. The obtained results including the maximum absolute error for $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ are given in Table 2.

Table 2: Maximum absolute error of LWM for $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ of problem 5.2.

<table>
<thead>
<tr>
<th>$M$</th>
<th>Max.err of $x_1(t)$</th>
<th>Max.err of $x_2(t)$</th>
<th>Max.err of $x_3(t)$</th>
<th>Max.err of $x_4(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$2.26\times10^{-6}$</td>
<td>$6.72\times10^{-5}$</td>
<td>$3.89\times10^{-6}$</td>
<td>$7.27\times10^{-5}$</td>
</tr>
<tr>
<td>12</td>
<td>$5.30\times10^{-8}$</td>
<td>$8.22\times10^{-7}$</td>
<td>$7.24\times10^{-8}$</td>
<td>$7.13\times10^{-7}$</td>
</tr>
<tr>
<td>14</td>
<td>$1.01\times10^{-9}$</td>
<td>$3.68\times10^{-8}$</td>
<td>$9.63\times10^{-10}$</td>
<td>$3.84\times10^{-8}$</td>
</tr>
<tr>
<td>16</td>
<td>$1.65\times10^{-11}$</td>
<td>$5.62\times10^{-10}$</td>
<td>$7.93\times10^{-12}$</td>
<td>$5.46\times10^{-10}$</td>
</tr>
<tr>
<td>18</td>
<td>$4.03\times10^{-13}$</td>
<td>$1.06\times10^{-11}$</td>
<td>$1.75\times10^{-13}$</td>
<td>$1.06\times10^{-11}$</td>
</tr>
<tr>
<td>20</td>
<td>$3.98\times10^{-14}$</td>
<td>$2.51\times10^{-13}$</td>
<td>$1.52\times10^{-13}$</td>
<td>$1.62\times10^{-13}$</td>
</tr>
</tbody>
</table>

The convergency and capability of the LWM can be observed by a closer look at the results of Table 2. 

Conclusions:

The Legendre wavelets method was successfully applied for solving DAEs. Due to orthonormality of Legendre wavelets, this proposed method is easy to implement and yields desire accuracy only in a few terms.

REFERENCES