Polynomial Optimization with Application to Sensor Network Localization

1Mojtaba Dehghan Banadaki, 2Samira Smaeili

1Department of Mathematics, Yazd University, Yazd, Iran. 2Department of Mathematics, Tarbiat Modares University, Tehran, Iran.

Abstract: In this paper, a convexification method for computing the global minimum of a polynomial optimization problem (POP) is presented. In this method, using moments theory, slightly problem is changed to a sequence of linear convex optimization problems with linear matrix inequality (LMI) constraints, that convergence of this sequence to the global optimal value of the original problem is guaranteed. Finally, the sensor network localization problem is introduced and formulated as finding the global minimizer of a quartic polynomial.

Key words: Polynomial Optimization, Convex Optimization, Moment Matrix, Linear Matrix Inequality, Sensor Network Localization.

INTRODUCTION

Polynomial optimization problem (POP) is a nonlinear and generally non-convex programming that objective function and constraints are multivariate polynomials.

A vast number of design problems in control engineering and system theory can be posed as constrained POPs. The goal of solving these problems is to find their global minimums. In recent years, convex optimization has become a computational tool of central importance in engineering, thanks to its ability to solve very large, practical engineering problems reliably and efficiently (Hindi, H., 2004). Convex optimization problems have three crucial properties that make them fundamentally more tractable than generic non-convex optimization problems:

1. no local minima: any local optimum is necessarily a global optimum;
2. exact infeasibility detection: using duality theory, hence algorithms are easy to initialize;
3. efficient numerical solution methods that can handle very large problems (Hindi, H., 2004).

Therefore, according to this reality that any local solution is also global for convex optimization problems, we can change the POP to a convex optimization problem and solve it.

Methodology:

Consider the POP

\[ p^* = \min_x g_0(x) \]

\[ g_i(x) \geq 0 \quad i = 1, 2, ..., m \]

Where the mappings \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \), \( i = 0, 1, ..., m \) are real-valued polynomials, that is, \( g_i \in \mathbb{R}[x_1, ..., x_n] \), the ring of polynomials in n-variables. Depending on its parity, let \( g_i = 2d_i - 1 \) or \( 2d_i \), and denote \( d = \max_i d_i \). Define \( V_r(x) = \left[ 1, x_1, x_2, ..., x_n, x_1^2, x_1 x_2, ..., x_r \right] \) as a basis for the space of polynomials with degree at most k. A polynomial \( g \in \mathbb{R}[x_1, ..., x_n] \) can be written as

\[ x \mapsto g(x) = \sum_{\alpha_n \in \mathbb{N}^n} g_{\alpha_n} x^\alpha \quad \text{where} \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n} \quad \text{is a monomial of degree} \quad |\alpha| := \sum_{i=1}^n \alpha_i. \]

Let \( g = \{g_{\alpha}\} \in \mathbb{R}^{(r)} \) be the vector of coefficients of the polynomial \( g(x) \) in the basis (2) where

\[ s(r) := \binom{r + n}{r}. \]

Given a \( s(2r) \)-sequence \((1, y_1, ...)\), let \( M_r(y) \) be the moment matrix of dimension \( s(r) \) with rows and columns indexed by (2). The moment matrix \( M_r(y) \) is the block matrix \( \{M_{ij}(y)\}_{0 \leq i, j \leq r} \) defined by

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Corresponding Author: Mojtaba Dehghan Banadaki, Department of Mathematics, Yazd University, Yazd, Iran.
Where $y_{i,j}$ represents the $(i+j)$-order moment $\int x^iy^j \mu(d(x,y))$ for some probability measure $\mu$ [2].

To fix ideas, when $n=2$ and $m=3$, one obtains

$$M_2(y) = \begin{bmatrix}
1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\
y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\
y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\
y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\
y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\
y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4}
\end{bmatrix}.$$
For $\alpha \in \mathbb{N}^n$, $g_0(x) = \sum_{\alpha} (g_0)_{\alpha} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is a polynomial, and thus the criterion is a finite linear combination $\sum_{\alpha} (g_0)_{\alpha} y_\alpha$ of moments $y_\alpha = \int x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} d\mu$ of the probability measure $\mu$.

For $r \geq d$ we define for (generally non-convex) problem (1) a hierarchy $\{Q_r\}$ of (convex) LMI relaxations:

$$
\begin{align*}
Q_r: \quad & \min_y \sum_{\alpha} (g_0)_{\alpha} y_\alpha \\
& \text{s.t.} \quad M_r(y) \succeq 0 \\
& \quad M_{r-d_i}(g_i, y) \succeq 0, \quad i = 1, 2, \ldots, m
\end{align*}
$$

where each decision variable $y_\alpha$ of $y = \{y_\alpha\}$ corresponds to a monomial $x^\alpha$.

$M_r(y)$ is the positive semi-definite moment matrix of order $r$, and

$M_{r-d_i}(g_i, y)$ is the positive semi-definite localizing matrix of order $r - d_i$, associated with the polynomial $g_i$, for all $i = 1, 2, \ldots, m$.

Using the mild assumption 1 on the polynomials $\{g_i\}$, we have the following converging theorem.

**Theorem 1:**

Let $p(x) : \mathbb{R}^n \to \mathbb{R}$ be an $r$-degree polynomial and $K$ be a compact set. Let Assumption 1 hold, and let $p_K^* := \min_{x \in K} p(x)$.

Then as $r \to \infty$, one has $\inf Q_r \uparrow p_K^*$.

**Proof 2:**

**Numerical illustrations:**

In this section, we provide some examples to illustrate the efficiency of the method.

**General polynomial optimization problems:**

We use GloptiPoly software to solve the numerical examples. GloptiPoly is a Matlab/SeDuMi add-on to build and solve convex linear matrix inequality (LMI) relaxations of the (generally non-convex) global optimization problem of minimizing a multivariable polynomial function subject to polynomial inequality, equality or integer constraints (Henrion and Lasserre, 2003).

**Example 1:**

Consider the non-convex optimization problem

$$
\begin{align*}
\min & \quad x_2^2 \\
\text{s.t.} & \quad 4 - x_1^2 - x_2^2 \geq 0 \\
& \quad \frac{1}{2} x_1^2 + x_2 - 1_2 \geq 0
\end{align*}
$$

Where the objective function $x \mapsto x_2^2$ is minimized over a non-convex feasible set delimited by circular and parabola arcs. The feasible region is shown in Figure 1.

The first LMI relaxation $Q_1$ is
max \( y_{0,2} \)

\[
\begin{bmatrix}
1 & y_{1,0} & y_{0,1} \\
y_{1,0} & y_{2,0} & y_{1,1} \\
y_{0,1} & y_{1,1} & y_{0,2}
\end{bmatrix} \geq 0
\]

\[4 - 2y_{2,0} - y_{0,2} \geq 0\]

\[\frac{1}{2}y_{2,0} + y_{0,1} - 1 \geq 0\]

With optimal value \( p_1 = 1.4321 \times 10^{-13} \). In this relaxation, the \( 3 \times 3 \) positive semi-definite matrix is a moment matrix of order up to 2. Problem constraints are linearized with the help of these moment variables. GloptiPoly software reports \( status=0 \). This means that no globally optimal solution could be extracted in the first relaxation.

The second LMI relaxation \( Q_2 \) is

max \( y_{0,2} \)

\[
\begin{bmatrix}
1 & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\
y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\
y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\
y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\
y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\
y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4}
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
4 - 2y_{2,0} - y_{0,2} & 4y_{1,0} - y_{3,0} - y_{1,2} & 4y_{0,1} - y_{2,1} - y_{0,3} \\
4y_{1,0} - y_{3,0} - y_{1,2} & 4y_{2,0} - y_{4,0} - y_{2,2} & 4y_{1,1} - y_{3,1} - y_{1,3} \\
4y_{0,1} - y_{2,1} - y_{0,3} & 4y_{1,1} - y_{3,1} - y_{1,3} & 4y_{0,2} - y_{2,2} - y_{0,4}
\end{bmatrix} \geq 0
\]

\[
\begin{bmatrix}
\frac{1}{2}y_{2,0} + y_{0,1} - 1 & \frac{1}{2}y_{3,0} + y_{1,1} - y_{1,0} & \frac{1}{2}y_{2,1} + y_{0,2} - y_{0,1} \\
\frac{1}{2}y_{3,0} - y_{1,1} - y_{1,0} & \frac{1}{2}y_{4,0} + y_{2,1} - y_{2,0} & \frac{1}{2}y_{3,1} + y_{1,2} - y_{1,1} \\
\frac{1}{2}y_{2,1} + y_{0,2} - y_{0,1} & \frac{1}{2}y_{3,1} - y_{1,2} - y_{1,1} & \frac{1}{2}y_{2,2} + y_{0,3} - y_{0,2}
\end{bmatrix} \geq 0
\]

With optimal value \( p_2 = 2.3148 \times 10^{-10} \). This problem features a \( 6 \times 6 \) moment matrix corresponding to moments of order up to 4. GloptiPoly software reports \( status=1 \). This means that the global optimum is attained in this relaxation and is \( p^* = p_2 = 2.3148 \times 10^{-10} \). GloptiPoly software also gives us the global minimizers \( (x_1^*, x_2^*) = (-1.7676, 0.0000) \) and \( (x_1^*, x_2^*) = (1.7676, 0.0000) \).

**Application to sensor network localization:**

Recent advances in micro-electro-mechanical systems (MEMS) and wireless communication technology has made possible the large-scale deployment of wireless sensor networks with thousands of nodes. Some of the application areas for sensor networks are industrial automation (process control), military (real-time monitoring of troop movements), utilities (automated meter reading), building control and environmental monitoring (Srirangarajan et al., 2008).

Sensor network localization has been an area of active research in recent years with a large number applications (Srirangarajan et al., 2008). The goal of this problem is to determine the position of all sensor nodes in the network with knowledge of the positions of some nodes (called anchors) and some pairwise distance measurements.
Fig. 1: Feasible set for Example 1. The feasible set (dark region) is non-convex and delimited by circular and parabola arcs.

The localization problem is mathematically formulated as: Consider \( n \) distinct nodes in \( \mathbb{R}^d \) \((d \geq 1)\). Given the positions of the last \( (n-m) \) nodes (called anchors) \( x_{m+1}, \ldots, x_n \) and the Euclidean distance \( d_{ij} \) between neighboring nodes \( i \) and \( j \) where \((i,j) \in A\). \( A \) is the undirected neighbor set defined as:

\[
A := \{(i,j) : \|x_i - x_j\| \leq \text{RadioRange}\}
\] (Srirangarajan et al., 2008). We need to find the Cartesian coordinates of the first \( m \) nodes (called sensors). This can be formulated as the following global optimization (Srirangarajan et al., 2008):

\[
\min_{x_1,\ldots,x_m \in \mathbb{R}^d} \sum_{(i,j) \in A} \left\| x_i - x_j \right\|^2 - d_{ij}^2.
\] (3)

This optimization problem is non-convex and it is also NP-hard to find global solutions. So approximation methods are great interests. In particular, various convex relaxations have been proposed to approximate the problem. Examples include second-order cone programming (SOCP) relaxations (Tseng, 2007), semi-definite programming (SDP) relaxations (Kim et al., 2009), edge-based semi-definite programming (ESDP) relaxations (Tseng, 2007) and sum of squares (SOS) relaxations (Kim et al., 2009).

In problem (3), the objective is a sum of absolute values, and hence not a polynomial. However, if we replace the absolute values by squares, we can get a new optimization problem

\[
\min_{x_1,\ldots,x_m \in \mathbb{R}^d} \sum_{(i,j) \in A} \left( x_i - x_j \right)^2 - d_{ij}^2.
\] (4)

Such that the objective function is a polynomial of degree four. We next present an example to illustrate the LMI relaxations for problem (4).

Example 2: Consider a sensor network problem in \( \mathbb{R}^2 \) as described in Figure 2, with two sensors and two anchors \( x_1 = (a,b) \), \( x_2 = (c,d) \), \((-1,1)\) and \((1,2)\) respectively. The network is as \( A = \{(1,2),(1,3),(1,4),(2,4)\} \) and distances are given by \( d_{13} = 2 \), \( d_{14} = 3 \) and \( d_{12} = d_{24} = 1 \). For this example, we can formulate it as a polynomial optimization and then apply LMI relaxations to solve it. Therefore, problem (4) now becomes

\[
\min_{x_1, x_2 \in \mathbb{R}^2} \left( (a-c)^2 + (b-d)^2 - 1 \right)^2 + \left( (a+1)^2 + (b-1)^2 - 4 \right)^2 + \left( (a-1)^2 + (b-2)^2 - 9 \right)^2 + \left( (c-1)^2 + (d-2)^2 - 1 \right)^2.
\]
Using GloptiPoly software, we can find the optimal solution as 2.7778 in the first LMI relaxation and two global minimizers $x_1^* = (0.0038, -0.7298)$ and $x_2^* = (0.5019, 0.6351)$ as two sensors Cartesian coordinates.

Conclusion:

In this paper, we presented a method for the computation of global minimum in a polynomial optimization problem using linear matrix inequality (LMI) relaxations that the solution of this relaxation is convergence to the exact solution of original problem.

The main ideas in this method are as follow:

1. Increasing additional inequality constraints to a optimization problem in Euclidean space $R^n$.
2. Projecting the original problem with additional constraints in $R^n$ onto an equivalent optimization problem in a matrix space.
3. Relaxing the polynomial constraints such that the feasible set be convex.
4. Projecting the relaxed problem from matrix space onto Euclidean space $R^n$.

Finally, we have referred to the applications of this method in general polynomial optimization problems and sensor network localization, and presented some examples. If we have a small number of sensors, Gloptipoly works very well.

REFERENCES


