An Algorithm to Solve Fully Fuzzy Biobjective Linear Programming Based on the Compromise Programming with Respect to the Ideal Points

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Abstract: In this paper, a fully fuzzy biobjective linear programming (FFBOLP) problem will be considered, which all the parameters and variables are trapezoidal fuzzy numbers. In order to solve the FFBOLP, a compromise programming and a ranking function, which introduced by Yager [25], will be used to reach our goal. In fact, we are going to seek a fuzzy feasible solution as close as possible to the ideal points. Finally, a numerical example is displayed to illustrate the proposed method.

Key words: fuzzy numbers; fully fuzzy biobjective linear programming; crisp biobjective linear programming (CBOLP); Compromise programming.

INTRODUCTION

The fuzzy multiobjective linear programming (FMOLP) problems involving fuzzy parameters would be viewed as a more realistic version than the conventional one (Sakawa.). Various kinds of FMOLP models have been proposed to deal with different decision-making situations that involve fuzzy values in objective function parameters, constraints parameters, or goals.

In 1978, Zimmerman, extended his fuzzy linear programming approach to the multiobjective linear programming problem. Tanaka and Asai (1984a) formulated the FMOLP with triangular fuzzy numbers, and the nonlinear programming problem obtained was solved by using a max-min operator. Rummelfanger et al., (1989) presented a new method called “alpha-level related pair formation” for solving linear programming problems with fuzzy parameters in the objective function. Fullér, (1993) studied the fuzzy linear programming (FLP) problems with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes (MFR), where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem. Fedrizzi (Fuller, 1994) explored stability analysis in possibilistic programming by extending previous research results to possibilistic linear programs with multiobjective functions. Kahraman et al., (1996) proposed a fuzzy multiobjective linear programming that considers intangible benefits in AMTs and expands the constraints by adding tolerances. Borges and Antunes, (2003) studied the effects of uncertainty on multiobjective linear programming models using the concepts of fuzzy set theory. The proposed interactive decision support system was based on the interactive exploration of the weight space. Jana and Roy, (2005) presented the solution procedure for a fuzzy multiobjective linear programming (FMOLP) problem with mixed constraints and its application in solid transportation problem. Wu, et al., (2006) developed a new approximate algorithm for solving FMOLP problems involving fuzzy parameters in any form of membership functions in both objective functions and constraints. Duran Toksan, (2008) presented the use of a Taylor series for fuzzy multiobjective linear fractional programming (FMOLF) problems. In the proposed approach, membership functions associated with each objective of fuzzy multiobjective linear fractional programming problem transformed using a Taylor series were unified. Ammar, (2009) considered a multiobjective quadratic programming problem having fuzzy random coefficients matrix in the objective and constraints and the decision vector are fuzzy pseudo random variables. Zhang et al., (2010) surveyed a fuzzy-robust stochastic multiobjective programming (FRSMOP) approach, which integrates fuzzy-robust linear programming and stochastic linear programming into a general multiobjective programming framework. Ezzati, et al., solved a fully fuzzy linear programming using min-max method such that all parameters and variables in the model are triangular fuzzy numbers. A new method was proposed by Kumar, et al., (2011) to find the fuzzy optimal solution of fully fuzzy linear programming problems with equality constraints. By using the proposed method the fuzzy optimal solution of FFLP problems with equality constraints could be easily obtained.
In this paper, we are going to solve the FFBOLP problem such that all the parameters and variables are considered as the trapezoidal fuzzy numbers by use of Kumar's method, (2011). Firstly, a ranking function will be used, which proposed by Yager, to change the FFBOLP problem to the crisp biobjective linear programming (CBOLP) problem. Then, the compromise programming will be applied to find the (weakly) efficient solution as close as possible to the ideal points. Finally, a numerical example is displayed to illustrate the proposed method.

This paper is organized as follows: In section 2, some definitions and notations of fuzzy numbers and also definition of the ranking function are presented. In section 3, the FFBOLP problem and definition of the compromise solution are introduced. In section 4, the compromise programming will be used to seek the compromise solution of the FFBOLP. To illustrate the proposed method, a numerical example is solved in section 5. Conclusions are discussed in section 6.

2. Preliminaries:

Definition 1:
A fuzzy number is a fuzzy set like \( \tilde{u} : \mathbb{R} \rightarrow I = [0,1] \) which satisfies,
1. \( u \) is \(-\)continuous,
2. \( u(x) = 0 \) outside some interval \([c,d]\),
3. There are real numbers \( a, b \) such that \( c \leq a \leq b \leq d \) and
   3.1 \( u(x) \) is monotonic increasing on \([c,a]\),
   3.2 \( u(x) \) is monotonic decreasing on \([b,d]\),
   3.3 \( u(x) = 1, a \leq x \leq b \).

The set of all these fuzzy numbers is denoted by \( E \). An equivalent parametric from of that is presented as follows.

Definition 2:
A fuzzy number \( \tilde{u} \) parametric form is a pair \((\tilde{u}, \tilde{u})\) of functions \( u(r), \tilde{u}(r), 0 \leq r \leq 1 \), which satisfy the following requirements:
1. \( u(r) \) is a bounded monotonic increasing left continuous function,
2. \( \tilde{u}(r) \) is a bounded monotonic decreasing left continuous function,
3. \( u(r) \leq \tilde{u}(r), 0 \leq r \leq 1 \).

A popular fuzzy number is the trapezoidal fuzzy number \( \tilde{u} = (x_0, y_0, \alpha, \beta) \) with interval defuzzifier \([x_0,y_0]\) and left fuzziness \( \alpha \) and right fuzziness \( \beta \) where the membership function is:

\[
\begin{align*}
    u(x) &= \begin{cases} 
        \frac{1}{\alpha} (x - x_0 + \alpha) & \text{if } x_0 - \alpha \leq x \leq x_0, \\
        1 & \text{if } x \in [x_0,y_0], \\
        \frac{1}{\beta} (y_0 - x + \beta) & \text{if } y_0 \leq x \leq y_0 + \beta, \\
        0 & \text{Otherwise}.
    \end{cases}
\end{align*}
\]

Its parametric form is \( u(r) = x_0 - \alpha + \alpha r, \tilde{u}(r) = y_0 + \beta + \beta r \).

Definition 3:
(Liou, 1992) A trapezoidal fuzzy numbers \( \tilde{u} = (x_1, y_1, \alpha_1, \beta_1)\) is said to be non-negative fuzzy number if and only if \( x_1 - \alpha_1 \geq 0 \).
Definition 4:
(Kaufmann, 1985) Let $TF(R)$ be the set of all trapezoidal fuzzy numbers. The arithmetic operations between two trapezoidal fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

Let $\tilde{u} = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{v} = (x_2, y_2, \alpha_2, \beta_2)$ be two trapezoidal fuzzy numbers and $K \in \mathbb{R}$. Define:

(i) $K \geq 0$ $k\tilde{u} = (kx_1, ky_1, k\alpha_1, k\beta_1)$.

(ii) $K < 0$ $k\tilde{u} = (kx_1, ky_1, -k\beta_1, -k\alpha_1)$.

(iii) $\tilde{u} \oplus \tilde{v} = (x_1 + x_2, y_1 + y_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$.

(iv) $\tilde{u} \ominus \tilde{v} = (x_1 - x_2, y_1 - y_2, \alpha_1 + \beta_2, \beta_1 + \alpha_2)$.

(v) $\tilde{u} = (x_1, y_1, \alpha_1, \beta_1)$ be any trapezoidal fuzzy number and $\tilde{v} = (x_2, y_2, \alpha_2, \beta_2)$ be a non-negative trapezoidal fuzzy number then

$$\tilde{u} \otimes \tilde{v} = \begin{cases} 
(x_1x_2, y_1, y_2, x_1 - (x_1 - \alpha_1)(x_2 - \alpha_2), (y_1 + \beta_2)(y_2 + \beta_2) - y_1y_2) & \text{if } x_1 - \alpha_1 \geq 0, \\
(x_1x_2, y_1, y_2, x_1 - (x_1 - \alpha_1)(y_2 + \beta_2), (y_1 + \beta_2)(y_2 + \beta_2) - y_1y_2) & \text{if } x_1 - \alpha_1 < 0, y_1 + \beta_2 \geq 0, \\
(x_1x_2, y_1, y_2, x_1 - (x_1 - \alpha_1)(y_2 + \beta_2), (y_1 + \beta_2)(x_2 - \alpha_2) - y_1y_2) & \text{if } y_1 + \beta_1 < 0.
\end{cases}$$

Up to now, many types of ranking functions have been introduced and some of them have been applied for solving linear programming problems with fuzzy parameters. Now, an effective approach for ranking the elements of $TF(R)$ will be defined.

Definition 5:
(Yager, 1981) A ranking function is a function $\mathbb{R} : TF(R) \rightarrow \mathbb{R}$, which maps each trapezoidal fuzzy number into the real line, where a natural order exists. Let $\tilde{u} = (x_0, y_0, \alpha, \beta)$ be a trapezoidal fuzzy number, then $\mathbb{R}(u) = \frac{1}{2}(x_0 + y_0 + \frac{1}{2}(\beta - \alpha))$ (the proposed by (Yager, 1981)).

Thus, for trapezoidal fuzzy numbers $\tilde{u} = (x_1, y_1, \alpha_1, \beta_1)$ and $\tilde{v} = (x_2, y_2, \alpha_2, \beta_2)$, we have: $\tilde{u} \asymp \tilde{v}$ if and only if $\frac{1}{2}(x_1 + y_1 + \frac{1}{2}(\beta_1 - \alpha_1)) \geq \frac{1}{2}(x_2 + y_2 + \frac{1}{2}(\beta_2 - \alpha_2))$.

3. Fully Fuzzy Biobjective Linear Programming:
A biobjective linear programming problem simultaneously optimizes two objectives subject to the given constraints. In general, the problem has no optimal solution that could optimize all objectives simultaneously.

The concept of optimal solution gives rise to the concept of nondominated solutions, for which no improvement in any objective function is possible without sacrificing at least one of the other objective functions.

We consider the following biobjective linear programming problem:

$$\max (c^1)^T x, (c^2)^T x)$$

s.t. $Ax = b$, $x \geq 0$, $x \geq 0$, $x \geq 0$,
where, parameters $c^i \in \mathbb{R}^n$, for $i = 1, 2$ are the objective functions, $A \in \mathbb{R}^{m \times n}$ is the constraint matrix, $b \in \mathbb{R}^m$ is the right hand side vector and $x \in \mathbb{R}^n$ is a vector of variables. The superscript $T$ over a vector or matrix denotes the transpose of the corresponding vector or matrix. We shall denote the feasible set of the MOLP by $X$. In the following we assume, without loss of generality, that $X$ is non empty. The objective function can be written as $C^T x$, where $C \in \mathbb{R}^{m \times 2}$ has columns $c^i$. An optimal solution $(x^* \in X)$ to this problem is (weakly) efficient if there is no $x \in X$ such that $C^T x^* \preceq C^T x$ and $C^T x^* \neq C^T x$ ($C^T x^* < C^T x$).

We can also write the compromise programming problems with respect to the ideal solution as follows:

$$\begin{aligned}
\text{min} \quad & d(f(x), y^i) \\
\text{s.t.} \quad & x \in X,
\end{aligned}$$

(1)

where $X = \{x : Ax = b, x \geq 0\}$ and $f(x) = ((c^1)^T x, (c^2)^T x)$ and $y^i = (y^i_1, y^i_2)$ is obtained as follows:

$$y^i_j = \max (c^i)^T x$$

s.t. $x \in X$, for $i = 1, 2$

In this paper, we will only consider metrics derived from norms as distance measures, i.e. 1, 2.

$\begin{aligned}
d(y^1, y^2) &= ||y^1 - y^2|| \\
\end{aligned}$

**Definition 6:**

(Ehrgott, 2005) 1. A norm $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}$ is called monotone, if $\| y^i \| \leq \| y^2 \|$ holds for all $y^i, y^2 \in \mathbb{R}^n$ with $|y^i_j| \leq |y^2_j|$ for $i = 1, 2, \ldots, n$ and moreover $\| y^i \| < \| y^2 \|$ if $|y^i_j| < |y^2_j|$ for $i = 1, 2, \ldots, n$.

2. A norm $\| \cdot \|$ is called strictly monotone, if $\| y^i \| < \| y^2 \|$ holds whenever $|y^i_j| \leq |y^2_j|$ for $i = 1, 2, \ldots, n$ and $|y^i_j| \neq |y^2_j|$ for some $j$.

**Theorem 1:**

(Ehrgott, 2005) 1. If $\| \cdot \|$ is monotone and $X^*$ is an optimal solution of (1) then $X^*$ is weakly efficient. If $X^*$ is a unique optimal solution of (1) then $X^*$ is efficient.

2. If $\| \cdot \|$ is strictly monotone and $X^*$ is an optimal solution of (1) then $X^*$ is efficient.

The most important class of norms is the class of $l_p - norm \| \cdot \| = \| \cdot \|_p$, i.e. $\| y \|_p = \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}$ for $1 \leq P < \infty$. Therefore, problem (1) may be shown, in terms of the $l_p - norm$, as follows:

$$\begin{aligned}
\min \left( \sum_{i=1}^{2} (y^i_j - (c^i)^T x) \right)^{\frac{1}{p}} \\
\text{s.t.} \quad & x \in X
\end{aligned}$$

(2)

The $l_p - norm \| \cdot \|_p$ is strictly monotone for $1 \leq P < \infty$ and monotone for $P = \infty$. The special cases $p = 1$ with $\| y \| = \left( \sum_{i=1}^{n} |y_i| \right)$ and $p = \infty$ with $\| y \| = \max |y_i|$ for $i = 1, 2, \ldots, n$ are of major importance.
As long as we just minimize the distance between a feasible point in objective space and the ideal point, we will find one (weakly) efficient solution for each choice of a norm.

Now, the FFBOLP problem may be formulated as follows:

$$\text{max} (\langle \tilde{c}^1 \rangle^T \otimes \tilde{x}, \langle \tilde{c}^2 \rangle^T \otimes \tilde{x})$$

s.t. \( \tilde{A} \otimes \tilde{x} \sim \tilde{b} \),
\( \tilde{x} \geq \tilde{0} \)

where all the parameters and variables belong to the set of fuzzy numbers. In the next section, an algorithm is going to be proposed to determine a set of the fuzzy compromise solutions of the FFBOLP, with respect to the ideal point, which is shown by \( \tilde{x}^* \).

### 4. The fuzzy compromise solution of the FFBOLP problem:

The best possible result of a multicriteria problem would be the ideal point \( y^I \). We know that when the objectives are conflicting the ideal values are impossible to obtain. So, the ideal point can use as a reference point, with the goal to search for solutions as close as possible to the ideal point. This is the basic idea of compromise programming.

In this section, the compromise programming is proposed to find the solution of following type of the FFBOLP problems:

$$\text{max} (\langle \tilde{c}^1 \rangle^T \otimes \tilde{x}, \langle \tilde{c}^2 \rangle^T \otimes \tilde{x})$$

s.t. \( \tilde{A} \otimes \tilde{x} \sim \tilde{b} \),
\( \tilde{x} \sim \tilde{0} \),

where all the variables and parameters of the model are trapezoidal fuzzy numbers and \( \tilde{x} \) is a non-negative trapezoidal fuzzy number.

The steps of the proposed method are as follows:

#### Step 1:
Substituting \( \tilde{C}_j = [\tilde{c}^{ji}]_{i=1}^n \), \( \tilde{x} = [\tilde{x}_j]_{j=1}^n \), \( \tilde{A} = [\tilde{a}_{ij}]_{m \times n} \) and \( \tilde{b} = [\tilde{b}_i]_{m \times 1} \) into the FFBOLP problem, then the FFBOLP may be written as:

$$\text{max} \left( \sum_{j=1}^{n} \tilde{c}^{1j} \otimes \tilde{x}_j, \sum_{j=1}^{n} \tilde{c}^{2j} \otimes \tilde{x}_j \right)$$

s.t. \( \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \sim \tilde{b}_i \), \( \forall i = 1, 2, ..., m \),
\( \tilde{x}_j \geq \tilde{0} \), \( \forall j = 1, 2, ..., n \).

#### Step 2:
If all the parameters and variables \( \tilde{c}^i_j \) \( (i=1,2) \), \( \tilde{x}_j \), \( \tilde{a}_j \) and \( \tilde{b}_j \) have been represented by trapezoidal fuzzy numbers \( (p^i_j, q^i_j, r^i_j, s^i_j) \) \( (i=1,2) \), \( (x_j, y_j, w_j, z_j) \) \( (a_j, b_j, c_j, d_j) \) and \( (b_j, g_j, h_j, k_j) \) respectively, then
the FFBOLP problem may be written as:

\[
\begin{align*}
\max & \quad \left( \sum_{j=1}^{n} (p^1_j, q^1_j, r^1_j, s^1_j) \otimes (x_j, y_j, w_j, z_j) \right) \leq \left( \sum_{j=1}^{n} (p^2_j, q^2_j, r^2_j, s^2_j) \otimes (x_j, y_j, w_j, z_j) \right) \\
\text{s.t.} & \quad \sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}, d_{ij}) \otimes (x_j, y_j, w_j, z_j) \succeq (b_i, g_i, h_i, k_i), \quad \forall i = 1, 2, \ldots, m, \\
& \quad (x_j, y_j, w_j, z_j) \succeq 0, \quad \forall j = 1, 2, \ldots, n.
\end{align*}
\]

**Step 3:**
Assuming \((a_{ij}, b_{ij}, c_{ij}, d_{ij}) = (m_{ij}, n_{ij}, o_{ij}, p_{ij})\) the FFBOLP obtained in step 2, may be written as:

\[
\begin{align*}
\max & \quad \Re\left( \sum_{j=1}^{n} (p^1_j, q^1_j, r^1_j, s^1_j) \otimes (x_j, y_j, w_j, z_j) \right), \Re\left( \sum_{j=1}^{n} (p^2_j, q^2_j, r^2_j, s^2_j) \otimes (x_j, y_j, w_j, z_j) \right) \\
\text{s.t.} & \quad \sum_{j=1}^{n} (m_{ij}, n_{ij}, o_{ij}, p_{ij}) \otimes (b_i, g_i, h_i, k_i), \quad \forall i = 1, 2, \ldots, m, \\
& \quad (x_j, y_j, w_j, z_j) \succeq 0, \quad \forall j = 1, 2, \ldots, n.
\end{align*}
\]

**Step 4:**
Using arithmetic operations, defined in the previous sections, the FFBOLP obtained in step 3, is converted into the following (CBOLP) problem:

\[
\begin{align*}
\max & \quad \Re\left( \sum_{j=1}^{n} (p^1_j, q^1_j, r^1_j, s^1_j) \otimes (x_j, y_j, w_j, z_j) \right), \Re\left( \sum_{j=1}^{n} (p^2_j, q^2_j, r^2_j, s^2_j) \otimes (x_j, y_j, w_j, z_j) \right) \\
\text{s.t.} & \quad \sum_{j=1}^{n} m_{ij} = b_i, \quad \forall i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} n_{ij} = g_i, \quad \forall i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} o_{ij} = h_i, \quad \forall i = 1, 2, \ldots, m, \\
& \quad \sum_{j=1}^{n} p_{ij} = k_i, \quad \forall i = 1, 2, \ldots, m, \\
& \quad (x_j, y_j, w_j, z_j) \succeq 0, \quad \forall j = 1, 2, \ldots, n.
\end{align*}
\]

**Step 5:**
To solve the CBOLP problem, we need to obtain the optimal objective values of the following problems:

\[
\begin{align*}
\max & \quad \Re\left( \sum_{j=1}^{n} (p^1_j, q^1_j, r^1_j, s^1_j) \otimes (x_j, y_j, w_j, z_j) \right)
\end{align*}
\]
\begin{align*}
\text{s.t. } & \sum_{j=1}^{n} m_{ij} = b_i, \quad \forall i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} n_{ij} = g_i, \quad \forall i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} o_{ij} = h_i, \quad \forall i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} p_{ij} = k_i, \quad \forall i = 1, 2, \ldots, m,
\end{align*}

for \( i = 1, 2 \). The optimal objective values of the problems are denoted by \( y^i_{1} \) for \( i = 1, 2 \). In fact, \( y^1_{1} \) and \( y^2_{1} \) are the ideal points correspond to the first and second objective functions.

**Step 6:**
Using the optimal objective values, that obtained in step 5 (i.e. \( y^i_{1} \) for \( i = 1, 2 \)) and model (2), we solve the following problem:

\[
\min \left\{ \frac{\sum_{i=1}^{2} \left( y^i_{1} - \mathcal{H}\left( \sum_{j=1}^{n} (p_{ij}^e, q_{ij}^e, r_{ij}^e, s_{ij}^e) \otimes (x_j, y_j, w_j, z_j) \right) \right) \right\}^{\frac{1}{p}}
\]

\[
\text{s.t. } \sum_{j=1}^{n} m_{ij} = b_i, \quad \forall i = 1, 2, \ldots, m, \\
\sum_{j=1}^{n} n_{ij} = g_i, \quad \forall i = 1, 2, \ldots, m, \\
\sum_{j=1}^{n} o_{ij} = h_i, \quad \forall i = 1, 2, \ldots, m, \\
\sum_{j=1}^{n} p_{ij} = k_i, \quad \forall i = 1, 2, \ldots, m,
\]

\( (x_j, y_j, w_j, z_j) \geq \tilde{0}, \quad \forall j = 1, 2, \ldots, n. \)

Note that, in order to scale the objective functions, each of them are divided by corresponding the ideal points i.e. \( y^1_{1} \) and \( y^2_{1} \).

**Step 7:**
Find the compromise solution \( x_j, y_j, w_j \) and \( z_j \) by solving the CBOLP problem obtained in step 6.

**Step 8:** Find the fuzzy compromise solution by putting the values of \( x_j, y_j, w_j \) and \( z_j \) in

\( \tilde{x}_j = (x_j, y_j, w_j, z_j). \)

**Step 9:**
Find the fuzzy compromise objective values by putting $\tilde{x}_j$ in $(\sum_{j=1}^{n} \tilde{c}_j^1 \otimes \tilde{x}_j, \sum_{j=1}^{n} \tilde{c}_j^2 \otimes \tilde{x}_j)$.

5 Numerical Example:

In this section the proposed method is illustrated with the numerical example.

Example 1:

Let us consider the following FFBOLP problem and solve it by the proposed method:

$$\text{max} \ (5, 8, 2, 5) \otimes \tilde{x}_1 \oplus (6, 10, 2, 6) \otimes \tilde{x}_2$$

$$\text{max} \ (1, 3, 1, 1) \otimes \tilde{x}_1 \oplus (2, 4, 1, 1) \otimes \tilde{x}_2$$

s.t. \((1, 1, 1, 1) \otimes \tilde{x}_1 \oplus (2, 2, 1, 1) \otimes \tilde{x}_2 \oplus (1, 1, 0, 0) \otimes \tilde{s}_1 \, \cup (10, 10, 9, 17), \) \(2, 2, 1, 1) \otimes \tilde{x}_1 \oplus (1, 1, 1, 1) \otimes \tilde{x}_2 \oplus (1, 1, 0, 0) \otimes \tilde{s}_2 \, \cup (11, 11, 7, 17), \)

\(\tilde{x}_j \geq \tilde{0}, \tilde{s}_j \geq 0, \quad \forall j = 1, 2,\)

where \(\tilde{x}_1 = (x_1, y_1, \alpha_1, \beta_1), \ \tilde{x}_2 = (x_2, y_2, \alpha_2, \beta_2), \ \tilde{s}_1 = (s_1, t_1, u_1, v_1)\) and \(\tilde{s}_2 = (s_2, t_2, u_2, v_2)\) are non-negative trapezoidal fuzzy numbers.

Using the arithmetic operations, defined in section 2 and definition 3, we have:

$$\text{max} \ 9(5x_1 + 6x_2, y_1 + 10y_2, 2x_1 + 3\alpha_1 + 2x_2 + 4\alpha_2, 5y_1 + 13\beta_1 + 6y_2 + 16\beta_2)$$

$$\text{max} \ 9(x_1 + 2x_2, 3y_1 + 4y_2, x_1 + \alpha_1 + \alpha_2, y_1 + 4\beta_1 + y_2 + 5\beta_2)$$

s.t. \((x_1 + 2x_2 + s_1, y_1 + 2y_2 + t_1, x_1 + x_2 + \alpha_1 + \alpha_2 + u_1, y_1 + 2\beta_1 + y_2 + 3\beta_2 + v_1) \, \cup (10, 10, 9, 17), \) \((x_2 + 2x_1 + s_2, y_2 + 2y_1 + t_2, x_2 + x_1 + \alpha_1 + u_2, y_2 + 2\beta_2 + y_1 + 3\beta_1 + v_2) \, \cup (11, 11, 7, 17), \)

\(\tilde{x}_j \geq \tilde{0}, \tilde{s}_j \geq 0, \quad \forall j = 1, 2.\)

Applying definition 5 and step 4 of the proposed method, the above FFBOLP problem is converted into the following CBOLP problem:

$$\text{max} \ \frac{1}{4} (8x_1 + 10x_2 + 21y_1 + 26y_2 + 13\beta_1 + 16\beta_2 - 3\alpha_1 - 4\alpha_2)$$

$$\text{max} \ \frac{1}{4} (x_1 + 3x_2 + 5y_1 + 9y_2 + 4\beta_1 + 5\beta_2 - \alpha_2)$$

s.t. \(x_1 + 2x_2 + s_1 = 10,\)

\(y_1 + 2y_2 + t_1 = 10,\)

\(y_1 + 2\beta_1 + y_2 + 3\beta_2 + v_1 = 17,\)

\(x_2 + 2x_1 + s_2 = 11,\)

\(y_2 + 2y_1 + t_2 = 11,\)

\(x_2 + x_1 + \alpha_1 + u_2 = 7,\)

\(y_2 + 2\beta_2 + y_1 + 3\beta_1 + v_2 = 17,\)

\(x_j - \alpha_j \geq 0, s_j - u_j \geq 0, \quad \forall j = 1, 2,\)

\(y_j - x_j \geq 0, t_j - s_j \geq 0, \quad \forall j = 1, 2,\)

\(\alpha_j \geq 0, \beta_j \geq 0, u_j \geq 0, v_j \geq 0, \quad \forall j = 1, 2.\)

To solve the CBOLP problem, we first obtain optimal objective value of the following problem:
max \( \frac{1}{4} (8x_1 + 10x_2 + 21y_1 + 26y_2 + 13\beta_1 + 16\beta_2 - 3\alpha_1 - 4\alpha_2) \)

s.t.  
\( x_1 + 2x_2 + s_1 = 10, \)
\( y_1 + 2y_2 + t_1 = 10, \)
\( y_1 + 2y_2 + t_1 = 10, \)
\( y_1 + 2\beta_1 + y_2 + 3\beta_2 + v_1 = 17, \)
\( x_2 + 2x_1 + s_2 = 11, \)
\( y_2 + 2y_1 + t_2 = 11, \)
\( x_2 + x_1 + \alpha_1 + u_2 = 7, \)
\( y_2 + 2\beta_2 + y_1 + 3\beta_1 + v_2 = 17, \)
\( x_j - \alpha_j \geq 0, s_j - u_j \geq 0, \quad \forall j = 1, 2, \)
\( y_j - x_j \geq 0, t_j - s_j \geq 0, \quad \forall j = 1, 2, \)
\( \alpha_j \geq 0, \beta_j \geq 0, u_j \geq 0, v_j \geq 0, \quad \forall j = 1, 2, \)

where, the optimal objective problem is led to \( y_1^f = 68.5. \) Also, we need to determine the ideal point corresponding to the second objective function, i.e. \( y_2^f \). Therefore, we need to solve the following problem:

max \( \frac{1}{4} (x_1 + 3x_2 + 5y_1 + 9y_2 + 4\beta_1 + 5\beta_2 - \alpha_2) \)

s.t.  
\( x_1 + 2x_2 + s_1 = 10, \)
\( y_1 + 2y_2 + t_1 = 10, \)
\( y_1 + 2\beta_1 + y_2 + 3\beta_2 + v_1 = 17, \)
\( x_2 + 2x_1 + s_2 = 11, \)
\( y_2 + 2y_1 + t_2 = 11, \)
\( x_2 + x_1 + \alpha_1 + u_2 = 7, \)
\( y_2 + 2\beta_2 + y_1 + 3\beta_1 + v_2 = 17, \)
\( x_j - \alpha_j \geq 0, s_j - u_j \geq 0, \quad \forall j = 1, 2, \)
\( y_j - x_j \geq 0, t_j - s_j \geq 0, \quad \forall j = 1, 2, \)
\( \alpha_j \geq 0, \beta_j \geq 0, u_j \geq 0, v_j \geq 0, \quad \forall j = 1, 2, \)

where, the optimal objective problem is led to \( y_2^f = 19.4. \)
Now, we solve problem (3) in terms of the $\ell_1 - norm$ and $\ell_\infty - norm$. The results are shown in Table 1. With respect to Table 1, we can conclude that the $\ell_\infty - norm$ is better than the $\ell_1 - norm$ because the objective value for $P = \infty$ is less than $P = 1$. It means that, the obtained result by the $\ell_\infty - norm$ is closer to the ideal points than the obtained result by the $\ell_1 - norm$. Besides, since the $\ell_1 - norm$ is strictly monotone, so, with respect to Theorem 1, the obtained results by $\ell_1 - norm$ are efficient solutions.

**Table 1:** Fuzzy compromise solutions with respect to the $\ell_p - norm$

<table>
<thead>
<tr>
<th>$l_1$ - norm</th>
<th>(4.0,2.0)</th>
<th>(3.3,2.2)</th>
<th>(0.0,0.0)</th>
<th>(0.0,0.0)</th>
<th>Objective value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_\infty$ - norm</td>
<td>(3.5,3,0.0,2.0,04)</td>
<td>(3.2,2.2)</td>
<td>(0.0,0.0)</td>
<td>(0.7,0.7,0.24,0)</td>
<td>(0.0180)</td>
</tr>
</tbody>
</table>

**Conclusion:**
In this paper, we have considered the FFBOLP. Because of the objective functions of the FFBOLP can be contradicted each others, therefore, we don't have a optimal solution for the FFBOLP. So, the compromise programming has been used for solving this problem to determine fuzzy compromise solutions of the problem, in terms of $l_1 - norm$ and $l_\infty - norm$ and ideal points, by use of the linear ranking function. Finally, a numerical example has been displayed to illustrate the proposed method.

**REFERENCES**


