Exact Solutions of (2+1) - Dimensional Nonlinear Evolution Equations by Using the Extended Tanh Method

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Abstract: In this paper, the extended tanh method is used to construct exact solutions of the generalized Kadomtsev-Petviashvili (gKP) equation in the form

\[ (u_t + 6u^nu_x + u_{xxx})_x + u_{yy} = 0, \quad n > 1, \]

and (2+1)-dimensional Schrödinger equation:

\[ iu_t + au_{xx} - bu_{yy} + c|u|^2u = 0. \]

The extended tanh method is an efficient method for obtaining exact solutions of nonlinear partial differential equations. This method can be applied to nonintegrable equations as well as to integrable ones.

Key words: Extended tanh method; Generalized Kadomtsev-Petviashvili (gKP) equation; (2+1)-dimensional Schrödinger equation.

INTRODUCTION

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as tanh and extended tanh methods (Malfliet, 1992; 2004; Wazwaz, 2001; 2005; 2008), (Taghizadeh and Mirzazadeh, 2010; Fan, 2000; El-Wakil and Abdou, 2007), hyperbolic function method (Xia, et al., 2001), sine-cosine method (Wazwaz, 2004; 2006) (Yusufoglu and Bekir, 2006), Jacobi elliptic function expansion method (Inc and Ergut, 2005), generalized Riccati equation method (Yan and Zhang, 2001), F-expansion method (Sheng, 2006), Hirota’s direct method (Hereman and Nuseir, 1997), and the first integral method (Feng, 2002; Ding and Li, 1996).

The aim of this paper is to find exact soliton solutions of the generalized Kadomtsev-Petviashvili (gKP) equation (Borhanifar et al., 2008), and the nonlinear (2+1)-dimensional Schrödinger equation (Zhou et al., 2004), by the extended tanh method.

The Extended Tanh Method and Tanh Method: a PDE

\[ F(u,u_x,u_t,u_{xx},u_{xt},u_{xxx},\ldots) = 0, \]

can be converted to an ODE

\[ G(u,u_x,u_t,u_{xx},u_{xt},\ldots) = 0, \]

upon using a wave variable \( \xi = x - St \). Eq. (2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.
The standard tanh method is developed by (Malfliet, 1992; 2004) where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. Introducing a new independent variable

\[ Y = \tanh(\mu \xi), \quad \xi = x - st, \quad (3) \]

leads to the change of derivatives:

\[ \frac{d}{d\xi} = \mu(1 - Y^2) \frac{d}{dY}, \]

\[ \frac{d^2}{d\xi^2} = -2\mu^2 Y(1 - Y^2) \frac{d}{dY} + \mu^2(1 - Y^2)^2 \frac{d^2}{dY^2}. \quad (4) \]

The extended tanh method admits the use of the finite expansion

\[ u(\mu \xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \quad (5) \]

where M is a positive integer, in most cases, that will be determined. Expansion (5) reduces to the standard tanh method for \( b_k = 0, \text{ for } k=1, \ldots, M \). Substituting (5) into the ODE (2) results in an algebraic equation in power of \( Y \).

To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. We then collect all coefficients of powers of \( Y \) in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters \( a_k \) (\( k=0, \ldots, M \)), \( b_k \) (\( k=1, \ldots, M \)), \( \mu \) and \( s \). Having determined these parameters we obtain an analytic solution \( u(x,t) \) in a closed form.

**Exact Solutions of GKP Equation:**

In this section we study the generalized KP equation

\[ (u_t + 6u^*u_x + u_{xxx})_x + u_{yy} = 0. \quad (6) \]

Using the variable \( u(x,y,t) = U(\mu \xi) \), \( \xi = k(x + ly - \lambda t) \), carries Eq. (6) into the ODE

\[ (l^2 - \lambda)U^* + 6nU^{n-1}(U^*)^2 + 6U^{n+1} + k^2 U^* = 0. \quad (7) \]

Twice integrating of Eq. (7), setting the constant of integrating to zero, we will have

\[ (l^2 - \lambda)U + \frac{6}{n+1} U^{n+1} + k^2 U^* = 0. \quad (8) \]

Balancing \( U^* \) with \( U^{n+1} \) in Eq. (8) gives

\[ M + 2 = (n+1)M, \]

then

\[ M = \frac{2}{n}. \]

To get a closed form solution, \( M \) should be an integer. To achieve our goal, we use the transformation

\[ U(\mu \xi) = V^{\frac{1}{n}}(\mu \xi). \quad (9) \]

This transformation (9) will change Eq. (8) into the ODE

\[ n^2(n+1)(l^2 - \lambda)V^* + 6n^2 V^2 - k^2((n^2 - 1)(V^*)^2 - n(n+1)V V^*) = 0. \quad (10) \]
Balancing $Y^{\nu}$ with $Y^{\mu}$ in Eq. (10) gives

$$2M + 2 = 3M,$$

then

$$M = 2.$$  

In this case, the extended tanh method the form (5) admits the use of the finite expansion

$$U(\mu\xi) = S(Y) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \quad (11)$$

Substituting the form (11) into Eq. (10) and using (4), collecting the coefficients of $Y$ we obtain:

Coefficients of $Y^6$: $6n^2 a_1^3 - 2k^2 \mu^2 (5n - 2)(n + 1)a_2^2$.

Coefficients of $Y^5$: $18n^2 a_1 a_2^2 - 4k^2 \mu^2 (3n - 1)(n + 1)a_3 a_2$.

Coefficients of $Y^4$: $[n^2(n + 1)(l^2 - \lambda) + 8k^2 \mu^2 (2n - 1)(n + 1)]a_1^2 - 2k^2 \mu^2 n(n + 1)a_5 a_1^2 + 18n^2 (a_0 a_1^2 + a_1 a_2^2).$

Coefficients of $Y^3$: $2n^2(n + 1)(l^2 - \lambda) + 2k^2 \mu^2 (9n - 4)(n + 1)]a_1 a_2 - 2k^2 \mu^2 n(n + 1)a_6 a_1 a_2 - 2k^2 \mu^2 (3n - 2)(n + 1)a_5^2 +$ $6k^2 \mu^2 n(n + 1)a_6 a_1 + 18n^2 (a_0 a_1^2 + a_1 a_2^2 + a_2 a_3^2) + 36n^2 a_0 a_1 a_2 + 6n^2 a_3^2.$

Coefficients of $Y^2$: $2n^2(n + 1)(l^2 - \lambda) + 8k^2 \mu^2 n(n + 1)]a_1 a_2 - 2k^2 \mu^2 (n + 1)a_1 b_2 - 8k^2 \mu^2 (n + 1)a_2 b_2 + 18n^2 (a_0 a_1^2 + a_1 a_2^2 + a_2 a_3^2) + 36n^2 a_0 a_1 a_2.$

Coefficients of $Y^1$: $+[2n^2(n + 1)(l^2 - \lambda) + 2k^2 \mu^2 (n + 1)(n + 4)]a_1 b_2 - 4k^2 \mu^2 (n + 1)a_2 b_2 + 18n^2 (a_0 b_1 + a_1 a_2 b_1) + 36n^2 (a_0 a_2 b_1 + a_1 a_3 b_1).$

Coefficients of $Y^0$: $+[2n^2(n + 1)(l^2 - \lambda) + 4k^2 \mu^2 (n + 1)]a_1 b_2 - 2k^2 \mu^2 n(n + 1)a_2 a_2 + 16k^2 \mu^2 (n + 1)]a_1 b_2 - 2k^2 \mu^2 n(n + 1)a_2 a_2 +$ $[2n^2(n + 1)(l^2 - \lambda) + 16k^2 \mu^2 (n + 1)]a_1 b_2 - 2k^2 \mu^2 n(n + 1)a_2 a_2 +$ $n^2(n + 1)(l^2 - \lambda)a_0^2 - k^2 \mu^2 (n^2 - 1)(a_1^2 + b_1^2) + 6n^2 a_3^2 + 18n^2 (a_0 b_2 + a_2 b_1^2) + 36n^2 (a_0 a_1 b_2 + a_3 a_2 b_2).$

Coefficients of $Y^{-1}$: $+[2n^2(n + 1)(l^2 - \lambda) + 2k^2 \mu^2 (n + 1)(n + 4)]a_1 b_2 - 4k^2 \mu^2 (n + 1)a_2 b_2 + 18n^2 (a_0 b_1 + a_1 a_2 b_1) + 36n^2 (a_0 a_2 b_1 + a_1 a_3 b_1).$

\[ [n^2(n+1)(l^2-\lambda) + 2k^2\mu^2(2n-1)(n+1)]b_1^2 - 2k^2\mu^2(3n-2)(n+1)b_2^2 + 18n^2(a_0b_1^5 + a_0^2b_2 + a_0b_2^5) + 36n^2a_0b_2. \]

Coefficients of \( Y^{-2} \):
\[
[2n^2(n+1)(l^2-\lambda) + 8k^2\mu^2(n(n+1))a_0b_2 - 2k^2\mu^2(n+1)a_0b_1 - 8k^2\mu^2(n+1)a_0b_2 + 18n^2(a_0b_1^5 + a_0^2b_2 + a_0b_2^5) + 36n^2a_0b_2.
\]

Coefficients of \( Y^{-3} \):
\[
[2n^2(n+1)(l^2-\lambda) + 2k^2\mu^2(9n-4)(n+1)]b_1b_2 - 2k^2\mu^2(n+1)b_1 + 18n^2a_0b_2^2 + 36n^2a_0b_2 + 6n^2b_3.
\]

Coefficients of \( Y^{-4} \):
\[
[n^2(n+1)(l^2-\lambda) + 8k^2\mu^2(2n-1)(n+1)]b_2^2 - k^2\mu^2(3n-1)(n+1)b_3^2.
\]

Coefficients of \( Y^{-5} \):
\[
18n^2b_2b_3^2 - 4k^2\mu^2(3n-1)(n+1)b_3^2.
\]

Coefficients of \( Y^{-6} \):
\[
6n^2b_3^3 - 2k^2\mu^2(5n-2)(n+1)b_3^2.
\]

Setting these coefficients equal to zero, and solving the resulting system, by using Maple, we find the following sets of solutions:

\[ a_0 = \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad a_1 = 0, \quad a_2 = \frac{1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad b_1 = 0, \quad b_2 = 0 \]

\[ \mu = \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}}. \]  

(12)

\[ a_0 = \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = \frac{1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad \mu = \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}}. \]

(13)

\[ a_0 = \frac{-1}{24} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad a_1 = 0, \quad a_2 = \frac{1}{48} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad b_1 = 0 \]

(14)

\[ b_2 = \frac{1}{48} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1}, \quad \mu = \frac{n}{4k} \sqrt{\frac{l^2-\lambda}{2n-1}}. \]

Recall that \( \frac{1}{U} = V^a \).

The sets (12)-(14) give the solitons solutions

\[ u_1(x,y,t) = \left\{ \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} \right\} \sec h^2\left[ \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}} \right] x, \]

(15)

\[ u_2(x,y,t) = \left\{ \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} \right\} \csc h^2\left[ \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}} \right] x, \]

(16)
\[ u_3(x,y,t) = \frac{-1}{48} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} (2 - \tanh^2 \left[ \frac{n}{4k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] - \coth^2 \left[ \frac{n}{4k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] )^n. \]  

Where \( \xi = k(x+ly-\lambda t) \).

However for \( l^2-\lambda < 0 \), we obtain the travelling wave solutions

\[ u_4(x,y,t) = \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} \sec^2 \left[ \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] )^n, \]

\[ u_5(x,y,t) = \frac{-1}{12} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} \csc^2 \left[ \frac{n}{2k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] )^n, \]

\[ u_6(x,y,t) = \frac{-1}{48} \frac{(5n-2)(n+1)(l^2-\lambda)}{2n-1} (2 + \tan \left[ \frac{n}{4k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] + \cot \left[ \frac{n}{4k} \sqrt{\frac{l^2-\lambda}{2n-1}} \xi \right] )^n. \]

**The (2+1)-Dimensional Schrödinger Equation:**

In this section we study the (2+1)-dimensional Schrödinger equation

\[ iu_t + au_{xx} - bu_{yy} + cu^2 u = 0. \]  

We introduce the transformation

\[ u(x,y,t) = e^{i\theta} U(\mu \xi), \quad \theta = \alpha x + \beta y + \delta t, \quad \xi = x + y + \lambda t, \]

Where \( \alpha, \beta, \delta \) and \( \lambda \) are constants and \( U(\mu \xi) \) is real function. Substituting (22) into Eq. (21) we obtain ordinary differential equation (ODE) in the form:

\[ i e^{i\theta} (i\delta U(\mu \xi) + \lambda \frac{\partial U(\mu \xi)}{\partial \xi} + a e^{i\theta} (-\alpha^2 U(\mu \xi) + 2ai \frac{\partial U(\mu \xi)}{\partial \xi} + \frac{\partial^2 U(\mu \xi)}{\partial \xi^2}) \]

\[ -b e^{i\theta} (-\beta^2 U(\mu \xi) + 2bi \frac{\partial U(\mu \xi)}{\partial \xi} + \frac{\partial^2 U(\mu \xi)}{\partial \xi^2}) + c \left| e^{i\theta} U(\mu \xi) \right|^2 e^{i\theta} U(\mu \xi) = 0. \]

Thus, we have

\[ -\delta U(\mu \xi) + i\lambda \frac{\partial U(\mu \xi)}{\partial \xi} - \alpha^2 U(\mu \xi) + 2ai \frac{\partial U(\mu \xi)}{\partial \xi} + a \frac{\partial^2 U(\mu \xi)}{\partial \xi^2} \]

\[ +b \beta^2 U(\mu \xi) - 2b \beta i \frac{\partial U(\mu \xi)}{\partial \xi} - b \frac{\partial^2 U(\mu \xi)}{\partial \xi^2} + c(U(\mu \xi))^3 = 0. \]

Substituting the relation \( \lambda = 2(b \beta - a\alpha) \) into Eq. (24), then \( U(\mu \xi) \) satisfies the following ODE:
\[-(\delta + a\alpha^2 - b\beta^2)U(\mu\xi) + \left(a - b\right)\frac{\partial^2 U(\mu\xi)}{\partial\xi^2} + c(U(\mu\xi))^3 = 0.\]  \hspace{1cm} (25)

Balancing $U''$ with $U^3$ in (25) gives

\[M + 2 = 3M,
\]
so that

\[M = 1.\]

The extended tanh method (5) admits the use of the finite expansion

\[U(\mu\xi) = S(Y) = a_0 + a_1Y + \frac{b_1}{Y}.\]  \hspace{1cm} (26)

Substituting (26) into (25), collecting the coefficients of $Y^j \quad -3 \leq j \leq 3$ we obtain:

Coefficients of $Y^3$: $2(a - b)\mu^2a_1 + ca_1^3$.

Coefficients of $Y^2$: $3ca_0a_1^2$.

Coefficients of $Y^1$: $3cb_0a_1^2 - \left(\delta + a\alpha^2 - b\beta^2\right)a_1 - 2(a - b)\mu^2a_1 + 3ca_1^2$.

Coefficients of $Y^0$: $6cb_0a_0a_1 + ca_0^3 - \left(\delta + a\alpha^2 - b\beta^2\right)a_0$.

Coefficients of $Y^{-1}$: $3ca_0b_1^2 - \left(\delta + a\alpha^2 - b\beta^2\right)b_1 - 2(a - b)\mu^2b_1 + 3cb_0b_1$.

Coefficients of $Y^{-2}$: $3ca_0b_1^2$.

Coefficients of $Y^{-3}$: $2(a - b)\mu^2b_1 + cb_1^3$.

Setting these coefficients equal to zero, and solving system, by using Maple, we find the following set of solutions:

\[\alpha_0 = 0, \quad \alpha_1 = 0, \quad b_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{c}}, \quad \mu = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{2(a - b)}}.\]  \hspace{1cm} (27)

\[\alpha_0 = 0, \quad \alpha_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{c}}, \quad b_1 = 0, \quad \mu = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{2(a - b)}}.\]  \hspace{1cm} (28)

\[\alpha_0 = 0, \quad \alpha_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{2c}}, \quad b_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{2c}}, \quad \mu = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{4(a - b)}}.\]  \hspace{1cm} (29)

\[\alpha_0 = 0, \quad \alpha_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{4c}}, \quad b_1 = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{4c}}, \quad \mu = \pm\sqrt{\frac{\delta + a\alpha^2 - b\beta^2}{8(a - b)}}.\]  \hspace{1cm} (30)

The sets (27)-(30) give the following solutions
$$U_1(\mu \zeta) = \pm \sqrt{\frac{\delta + aa^2 - bb^2}{c}} \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{2(a-b)}} \zeta).$$

$$U_2(\mu \zeta) = \pm \sqrt{\frac{\delta + aa^2 - bb^2}{c}} \tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{2(a-b)}} \zeta).$$

$$U_3(\mu \zeta) = \pm \frac{\sqrt{\delta + aa^2 - bb^2}}{2c} [\tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{4(a-b)}} \zeta) + \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{4(a-b)}} \zeta)].$$

$$U_4(\mu \zeta) = \pm \frac{\sqrt{\delta + aa^2 - bb^2}}{4c} [\tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{8(a-b)}} \zeta) + \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{8(a-b)}} \zeta)].$$

By (22) and in (x,y,t)-variables we obtain the following exact solitons solutions.

$$u_1(x,y,t) = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{\delta + aa^2 - bb^2}{c}} \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{2(a-b)}} (x+y+2(b\beta - a\alpha)t)).$$

$$u_2(x,y,t) = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{\delta + aa^2 - bb^2}{c}} \tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{2(a-b)}} (x+y+2(b\beta - a\alpha)t)).$$

$$u_3(x,y,t) = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{\delta + aa^2 - bb^2}{2c}} [\tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{4(a-b)}} (x+y+2(b\beta - a\alpha)t))$$
$$+ \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{4(a-b)}} (x+y+2(b\beta - a\alpha)t))].$$

$$u_4(x,y,t) = \pm e^{i(\alpha x + \beta y + \delta t)} \sqrt{\frac{\delta + aa^2 - bb^2}{4c}} [\tanh(\sqrt{\frac{\delta + aa^2 - bb^2}{8(a-b)}} (x+y+2(b\beta - a\alpha)t))$$
$$+ \coth(\sqrt{\frac{\delta + aa^2 - bb^2}{8(a-b)}} (x+y+2(b\beta - a\alpha)t))].$$

**Conclusion:**
In this paper, the extended tanh method has been successfully applied to find the exact solutions for generalized Kadomtsev-Petviashvili (gKP) equation and the (2+1)-dimensional Schrödinger equation. Thus, we can say that the proposed method can be extended to solve the problem of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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