Numerical Solution of One-dimensional Advection-diffusion Equation Using Simultaneously Temporal and Spatial Weighted Parameters

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Abstract: The several numerical techniques have been developed and compared for solving the one-dimensional advection-diffusion equation with constant coefficients. These techniques are based on the finite difference methods (FDM). By changing the values of temporal and spatial weighted parameters, solutions are obtained for both explicit and implicit techniques such as FTCS, FTBCS, BTCS, BTBSCS and Crank–Nicholson schemes. Numerical solution is given for two special cases which have been dealt with in the literature and for which an analytical solution has been provided. Comparison of the results has confirmed that the Crank-Nicholson numerical approach matches successfully with the analytical solution while the other techniques result in some levels of discrepancy.

Key words: finite difference methods; advection–diffusion equation; spatial weight; temporal weight; explicit and implicit techniques

INTRODUCTION

The significant applications of advection–diffusion equation lie in fluid dynamics (Kumar, 1988), heat transfer (Isenberg, 1972) and mass transfer (Guvanasen, 1983). Various approaches are available for solving one-dimensional advection–diffusion partial differential equations. Analytical solutions (Dehghan, 2004; Noye, 1988; Noye, 1989; Sankaranarayanan, 1998) to solve engineering problems are highly desirable due to the elegant connection that becomes visible between physical and mathematical principles. But the analytical solution of these equations containing complex initial and boundary conditions are usually unavailable. Graphical methods, finite element methods and finite difference methods (Bear, 1990; Kinzelbach, 1986; Remson, 1971; Wang, 1982; Zheng, 1995) are other approaches for solve partial differential equations (PDEs). Because the analytical solution of partial differential equations containing complex initial and boundary conditions are very difficult, it seems that the finite difference methods are appropriate for solving these equations. Eq. (1) shows the mathematical form of one-dimensional advection–diffusion phenomenon.

\[
\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2}
\]

initial condition

\[c(x,0) = f(x) \quad 0 \leq x \leq L\] (2)

and boundary conditions

\[c(0,t) = g(t) \quad , \quad 0 \leq t \leq T\] (3)

\[c(L,t) = h(t) \quad 0 \leq t \leq T\] (4)

Where \(f, g\) and \(h\) are known functions. \(u\) and \(D\) are the speed of advection and diffusivity respectively.

The domain contain \(0 \leq x \leq L\) and \(0 \leq t \leq T\).

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By changing only the values of temporal $\phi$, and spatial $\theta$, weighted parameters, Eq. (1) can be solved by various finite difference methods in both explicit and implicit modes (Dehghan, 2004; Karahan, 2006; Karahan, 2007).

2. Numerical Solution:

The mesh of grid-lines are introduced as

$$x_i = i\Delta x \quad i = 0, 1, 2, \ldots, M$$ (5)

$$t_n = n\Delta t \quad n = 0, 1, 2, \ldots, N$$ (6)

The constant spatial and temporal grid-spacing are $\Delta x = \frac{L}{M}$ and $\Delta t = \frac{T}{N}$, respectively (Karahan, 2006).

Where $M$ denotes the total number of spatial grid-spacing and $N$ denotes the total number of temporal grid-spacing.

Consider the following approximations of the derivatives in the advection–diffusion equation which incorporate time and space weights $\phi$ and $\theta$ as follows:

$$\frac{\partial c}{\partial t} = \frac{c(i, n+1) - c(i, n)}{\Delta t}$$ (7)

$$u \frac{\partial c}{\partial x} = (1 - \phi) \left\{ \frac{u}{\Delta x} \left[ (1 - \theta)c(i, n) + \theta c(i + 1, n) - (1 - \theta)c(i - 1, n) - \theta c(i, n) \right] \right\}$$

$$+ \phi \left\{ \frac{u}{\Delta x} \left[ (1 - \theta)c(i, n + 1) + \theta c(i + 1, n + 1) - (1 - \theta)c(i - 1, n + 1) + \theta c(i, n + 1) \right] \right\}$$ (8)

$$D \frac{\partial^2 c}{\partial x^2} = (1 - \phi) \left\{ \frac{D}{\Delta x^2} \left[ c(i - 1, n) - 2c(i, n) + c(i + 1, n) \right] \right\}$$

$$+ \phi \left\{ \frac{D}{\Delta x^2} \left[ c(i - 1, n + 1) - 2c(i, n + 1) + c(i + 1, n + 1) \right] \right\}$$ (9)

where $\phi$ is a time weighting factor and $\theta$ is the spatial weighting factor. Substituting equations (7), (8) and (9) into Eq. (1) gives:

$$c(i, n + 1) = -1 \frac{1}{A_0} \left[ A_2 c(i, n) + A_2 c(i + 1, n) + A_2 c(i - 1, n) + A_2 c(i + 1, n + 1) + A_2 c(i + 1, n + 1) \right]$$ (10)

Where

$$A_0 = 1 + \phi \left[ a(1 - 2\theta) + 2\theta \right]$$ (11)

$$A_i = -1 + (1 - \phi) \left[ a(1 - 2\theta) + 2\theta \right]$$ (12)
$$A_2 = (1 - \phi) \left[ a\theta - s \right]$$  \hfill (13)

$$A_3 = - (1 - \phi) \left[ a(1 - \theta) + s \right]$$  \hfill (14)

$$A_4 = \phi \left[ a\theta - s \right]$$  \hfill (15)

$$A_5 = - \phi \left[ a(1 - \theta) + s \right]$$  \hfill (16)

and

$$a = u \frac{\Delta t}{\Delta x}$$  \hfill (17)

$$s = D \frac{\Delta t}{\Delta x^2}$$  \hfill (18)

The stability of (10) for explicit finite difference schemes may be investigated using the von Neumann method (Hindmarsh, 1984). Implicit finite difference schemes are unconditionally von Neumann stable. The application of this stability feature shows that Eq. (10) is stable if

$$a^2 \leq (1 - \phi) \left[ a(1 - 2\theta) + 2s \right] \leq 1$$  \hfill (19)

3. Explicit Finite Difference Schemes:

In explicit finite difference schemes, for specified values of grid-nodes in $n$ time levels, we could compute the value of grid-nodes in $n+1$ time levels, by solving a simple algebraic equation. A common feature of the explicit finite difference method is the restriction caused for the value of the time step due to the stability requirements. This restriction necessitates the choice of extremely small values for $\Delta t$ (Isenberg, 1972).

3.1. The FTCS Technique:

This scheme uses the forward-difference routine for the time-derivative and centered-difference routine for all spatial derivatives. Application of the technique on solving Eq. (10) for $\phi = 0$ and $\theta = 0.5$ is depicted below.

$$c(i,n+1) = \left( s + \frac{a}{2} \right) c(i-1,n) + \left( 1 - 2s \right) c(i,n) + \left( s - \frac{a}{2} \right) c(i+1,n)$$  \hfill (20)

Also by substituting the $\phi = 0$ and $\theta = 0.5$ in Eq. (19), we can obtain the stability condition of this scheme.

$$\frac{a^2}{2} \leq s \leq 0.5$$  \hfill (21)

3.2. The FTBSCS Technique:

This technique - also called the explicit Upwind - uses the forward-difference form for the time derivative, centered-difference forms for the diffusive derivatives and backward differences forms for the spatial derivatives in the advection terms. Application of the technique on solving Eq. (10) for $\phi = 0$ and $\theta = 0$ is depicted below.
\[ c(i, n+1) = (s + a) c(i-1, n) + (1 - 2s - c) c(i, n) + s c(i+1, n) \]  \hspace{1cm} (22)

Also by substituting the \( \phi = 0 \) and \( \theta = 0 \) in Eq. (19), we can obtain the stability condition of this scheme

\[ \frac{a^2 - a}{2} \leq s \leq \frac{1-a}{2} \]  \hspace{1cm} (23)

4. Implicit Finite Difference Schemes:

In implicit finite difference schemes, for evaluating one node in \( n+1 \) time level, we must know the value of grid-nodes that exist around it, in \( n \) and \( n+1 \) time levels. Therefore we must solve a system of algebraic equations for each time step. We can solve these systems using the iterative method. The main disadvantage of these techniques is the extensive amount of computer time consumed in determining the numerical solution compared to the explicit methods for the same selection of values of \( s \) and \( a \). One advantage of implicit techniques is that they are unconditionally von Neumann stable. Thus we are not restricted in selecting the size of time step \( \Delta t \) (Dehghan, 2004).

4.1. The BTCS Technique:

This scheme uses the backward-difference form for the time-derivative and centered-difference forms for all spatial derivatives. Application of the technique on solving Eq. (10) for \( \phi = 1 \) and \( \theta = 0.5 \) is depicted below.

\[ c(i, n+1) = \frac{1}{1 + 2s} \left[ \left( s + \frac{a}{2} \right) c(i-1, n+1) + c(i, n) + \left( s - \frac{a}{2} \right) c(i+1, n+1) \right] \]  \hspace{1cm} (24)

4.2. The BTBSCS Technique:

This technique - also called the implicit Upwind- utilizes the backward-difference form for the time derivative, centered-difference forms for the diffusive derivatives and backward difference forms for the spatial derivatives in the advection terms. Application of the technique on solving Eq. (10) for \( \phi = 1 \) and \( \theta = 0 \) is depicted below.

\[ c(i, n+1) = \frac{1}{1 + a + 2s} \left[ (s + a) c(i-1, n+1) + c(i, n) + s c(i+1, n+1) \right] \]  \hspace{1cm} (25)

4.3. The Crank-Nicolson Type Technique:

In the implicit Crank–Nicholson type technique we replace all spatial derivatives with the average of their values at the \( n \) and \( n+1 \) th time levels and then substitute centered-difference forms for all derivatives. Application of the technique on solving Eq. (10) for \( \phi = 0.5 \) and \( \theta = 0.5 \) is depicted below.

\[ c(i, n+1) = \frac{1}{4 + 4s} \left[ (a + 2s) c(i-1, n+1) - (a - 2s) c(i+1, n+1) + (a + 2s) c(i-1, n) \right] \left[ - (a - 2s) c(i+1, n) + (4 - 4s) c(i, n) \right] \]  \hspace{1cm} (26)

The formulas derived in above finite difference methods are applied to \( i = 1, 2, \ldots, M - 1 \).

5. Numerical Applications:

In order to test the numerical schemes developed for solving (1), a special problem for which an exact solution is available is required so that approximate results obtained using the numerical techniques may be
compared with an exact solution. In this section two examples which have the analytical solutions in literature (Dehghan, 2004; Sankaranarayanan, 1998), are solved numerically to test the used finite difference schemes.

**Example 1:**

The analytical solution of the one-dimensional advection–diffusion in a region bounded by $0 \leq x \leq 1$ and $0 \leq t \leq 1$ is taken from Ref. [4] and given as

$$c(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp \left[ -\frac{(x + 0.5 - t)^2}{(0.00125 + 0.04t)} \right]$$

In this example the values of various used parameters are

$$D = 0.01 \ m^2/s, \ u = 1 \ m/s, \ \Delta x = 0.02 \ m, \ \Delta t = 0.004 \ s$$

and the initial and boundary conditions are

$$c(x,0) = \exp \left[ -\frac{(x + 0.5)^2}{0.00125} \right]$$

$$c(0,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp \left[ -\frac{(0.5 - t)^2}{(0.00125 + 0.04t)} \right]$$

$$c(1,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp \left[ -\frac{(1.5 - t)^2}{(0.00125 + 0.04t)} \right]$$

Regarding the initial condition, we have an initial Gaussian pulse of unit height. Fig. 1, shows the numerical solution of Eq. (1) at $T = 1s$ utilizing five methods compared with analytical solution. Regarding Fig. 1, the Crank-Nicolson scheme establishes results closer to the analytical solution. BTCS scheme provides suitable results too, and BTBSCS provides worst approximations. Fig. 2, depicts the absolute errors associating various methods that were introduced in this paper at $T = 1s$. Maximum absolute error of Crank-Nicolson scheme tends to $3.15 \times 10^{-3}$. The error of this scheme in $x = 0.5 \ m$ and $T = 1s$ is $7.4 \times 10^{-4}$ and this value is reported to be $9.8 \times 10^{-4}$ in Ref. (Hindmarsh, 1984), and $3.9 \times 10^{-3}$ in Ref. (Dehghan, 2004).

**Example 2:**

The analytical solution for the one-dimensional advection–diffusion in a Gaussian pulse of unit height, centered at $x=1m$ in a region bounded by $0 \leq x \leq 9$ and $0 \leq t \leq 1.5$ is taken from Ref. (Sankaranarayanan, 1998) and given as

$$c(x,t) = \frac{1}{\sqrt{1+4t}} \exp \left[ -\frac{(x - x_0 - ut)^2}{D(1+4t)} \right]$$

Where $u$ is the velocity in the $x$ direction, $x_0$ is the center of the initial Gaussian pulse, $D$ is the diffusion coefficient in the $x$ direction and $t$ is the time coordinate. The values of various used parameter are
Fig. 1: Comparison of analytical and numerical solutions for Example 1.

Fig. 2: Absolute errors of various methods for Example 1.

\[ D = 0.005 \text{ m}^2/\text{s}, \quad u = 0.8 \text{ m/s}, \quad \Delta x = 0.02 \text{ m}, \quad \Delta t = 0.01 \text{ s} \]

The initial and boundary conditions could be obtained from Eq. (31).

\[ c(x,0) = \exp \left[ -\frac{(x-1)^2}{D} \right] \]

\[ c(0,t) = \frac{1}{\sqrt{1+4t}} \exp \left[ -\frac{(1-ut)^2}{D(1+4t)} \right] \]

\[ \Delta s = \Delta u \cdot \Delta t \]

\[ \sum \Delta s \]
\[ c(9,t) = \frac{1}{\sqrt{1+4t}} \exp \left[ \frac{(8-ut)^2}{D(1+4t)} \right] \] (34)

Fig. 3, compares the numerical and analytical solutions in example 2, and Fig. 4 depicts the absolute errors of various methods for this example.

As shown in these two figures, the Crank-Nicolson scheme represents the minimum errors, such as previous example. And maximum absolute error of this scheme is $6.2 \times 10^{-3}$.

Fig. 3: Comparison of analytical and numerical solutions for Example 2.

Fig. 4: Absolute errors of various methods for Example 2.

As shown in these two figures, the Crank-Nicolson scheme represents the minimum errors, such as previous example. And maximum absolute error of this scheme is $6.2 \times 10^{-3}$.

Some examples about advection-diffusion equations are given in Ref. (Badrot-Nico, 2007).

6. Conclusions:

In this article several numerical methods were applied to the one-dimensional advection–diffusion equation. The numerical schemes satisfy this model very good. As just the values of temporal and spatial weighted parameters are changed, the solutions could be determined for both explicit and implicit techniques such as FTCS, FTBSCS, BTCS, BTBSCS and Crank–Nicolson schemes. The implicit methods are unconditionally stable but they need extensive computer time for determining the numerical solution. The explicit methods must satisfy the von Neumann stability conditions. By comparing the various techniques for determining the
numerical solution, it was found that the Crank-Nicolson scheme has a very good agreement with analytical results. Since for implicit techniques, we must solve a system of algebraic equations, it seems that the iterative is a suitable method for solving these systems.

**Symbols:**

- $c$ concentration
- $u$ advection coefficient
- $D$ diffusivity
- $L$ length
- $T$ total time
- $i$ space counter
- $n$ time counter
- $\phi$ temporal weight
- $\theta$ spatial weight
- $\Delta t$ time step
- $\Delta x$ space step

**REFERENCES**


