A Characterization of Compact Operators by Orthogonality
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Abstract: In this paper we extend the usual notion of orthogonality to Banach spaces. Also we establish a characterization of compact operators on Banach spaces that admit orthonormal Schauder bases.

Key words: Orthogonality, Compact operator, Banach space.

INTRODUCTION

Throughout this paper \( K \) is the field of real or complex numbers, \( E \) is a Banach space over \( K \) and norm denoted by \( \| \cdot \| \), and \( (x_n) = (x_n)_1^n = (x_n)_m \) a finite or infinite sequence in \( E \), where either \( N \) is a positive integer and \( L = \{1, 2, \ldots, N\} \) or \( N = \infty \) and \( L = \{1, 2, \ldots\} \). For \( (\emptyset \neq J \subset L) \), the closure of the span of the set \( \{x_n : n \in J\} \) is denoted by \( [x_n : n \in J] \).

The notion of orthogonality goes a long way back in time. Usually this notion is associated with Hilbert spaces or, more generally, inner product spaces. Various extensions have been introduced through the decades. Thus, for instance, \( x \) is orthogonal to \( y \) in \( E \)

(a) in the sense of [1], if for every \( \alpha \in K \),
\[
\|x + \alpha y\| \geq \|x\|;
\]
(b) in the sense of [6], if for every \( \alpha \in K \),
\[
\|x + \alpha y\| = \|x - \alpha y\|;
\]
(c) in the isosceles sense [4], if
\[
\|x + y\| = \|x - y\|;
\]
(d) in the Pythagorean sense [4], if
\[
\|x - y\|^2 = \|x\|^2 + \|y\|^2;
\]
(e) in the sense of [8], if
\[
\frac{x}{\|x\|} + \frac{y}{\|y\|} = \frac{x - y}{\|x - y\|}.
\]

One of the natural and simple properties of orthogonality in a Hilbert space \( H \) that one would like to hold true in a Banach space is that \( x \) is orthogonal to \( y \) in \( H \) if and only if

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Clearly, in any Banach space, Eq. (1) is equivalent to
\[ \| \lambda x + \mu y \| = \| \lambda \| \| x + \mu y \|, \]
\( \forall \lambda, \mu \in K \)  

Hence, we introduce the following definition:

**Definition 1.1:**
A finite or infinite sequence \((x_n)_{n \in L}\) in a Banach space \(E\) is said to be orthogonal if
\[ \sum_{n \in L} a_n x_n = \sum_{n \in L} a_n x_n, \]
for each \( \sum_{n \in L} a_n x_n \in E \).

If, in addition, \( \| x_n \| = 1 \) for all \( n \in L \), then \((x_n)_{n \in L}\) is said to be orthonormal. We write \( x \perp y \) if \( x \) is orthogonal to \( y \).

It is clear from the definition that \((x_n)_{n \in L}\) is orthogonal in \( E \) if and only if \((x_n)_{n \in L}\) is orthogonal in \([x_n : n = 1, 2, \ldots, L]\).

Note that Definition 1.1 is an extension of the usual notion of orthogonality since in a Hilbert space \(H\), \( x \perp y \) in our sense, if and only if, \( \langle x, y \rangle = 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \(H\).

**Theorem 1.1:**
(Saidi, 2002) Given a sequence \((x_n)_{n \in L}\) in \(E\), the following are equivalent:

(i) The sequence \((x_n)_{n \in L}\) is orthogonal in \(E\).

(ii) For each pair of sequences \((b_n)_{n \in L}\) and \((c_n)_{n \in L}\) in \(K\) satisfying \( |b_n| = |c_n| \) for all \( n \in L \), \( \sum_{n \in L} c_n x_n \) converges, if and only if \( \sum_{n \in L} b_n x_n \) converges and, if both converge,
\[ \left\| \sum_{n \in L} b_n x_n \right\| = \left\| \sum_{n \in L} c_n x_n \right\|. \]

**Remark 1.1:**
(Jinchaum, 2009; Singer, 1957). Invertibility of \(A\) is a necessary condition for existing the above factorization.

**Characterization of Compact Operators:**
Let \(L(F,E)\) denote the set of bounded linear operators from the normed space \(F\) into the Banach space \(E\). It is known that if \(F\) and \(E\) are Hilbert spaces, then \(T \in L(F,E)\) is compact if and only if \(T\) is the limit in \(L(F,E)\) of a sequence of finite-rank operators (Conway, 1985). This gives a convenient and practical characterization of compact operators in Hilbert spaces. We show here that the same characterization still holds true for any Banach space \(E\) that admits an orthonormal Schauder basis and any normed space \(F\). More precisely, we have
Definition 2.1:
A linear transformation $T: H \rightarrow H$ is compact if $T(\text{ball } H)$ has compact closure in $H$.

Remark 2.1:
The set of compact operators from $H$ into $H$ is denoted by $L(H, H)$ i.e.
$L(H, H) = \{ \phi: H \rightarrow H | H \text{ is the Hilbert space and } \phi \text{ is a linear function} \}$

Remark 2.2:
Let $H$ be the Hilbert space,

- $B_0(H, H) \subseteq B(H, H)$
- $B_0(H, H)$ is a linear space and \( \{T_n\} \subseteq L_0(H, K) \) and $L_0(H, K)$ is such that $\|T_n - T\| \rightarrow 0$ then $T \in B_0(H, H)$.
- if $A \in B(H, H), B \in L(H, H)$ then $TA$ and $BT$ are belong to $B_0(H, H)$.

Remark 2.3:
If $T \in B(H, H)$, the following statements are equivalent

- $T$ is compact
- $T^*$ is compact
- there is a sequence \( \{T_n\} \) of operators of a rank such that $\|T - T_n\|_H \rightarrow 0$

Corollary 2.1:
If $T \in B(H, H)$, then $\text{cl} (\text{ran} T)$ is separable and if \( \{e_j\} \) is a basis for $\text{cl} (\text{ran} T)$ and $P_n$ is the projection of $H$ onto $V\{e_j : 1 \leq j \leq n\}$ then $\|P_n T - T\| \rightarrow 0$.

Example 2.1:
If $(X, \Omega, \mu)$ is a measurable space and $K \in \ell^2(X \times X, \Omega \times \Omega, \mu \times \mu)$ then
$$(Kf)(x) = \int k(x, y) f(y) d\mu(y) \text{ is a compact operator and } \|K\| \leq \|k\|_2.$$  

Theorem 2.1:
Suppose that \( \{e_n\}_{n=1}^\infty \) is an orthonormal Schauder basis of the Banach space $E$ and that $F$ is a normed space. For each positive integer $k$, let $P_k$ be the projection on \( \{e_n : 1 \leq n \leq k\} \) defined by
$$P_k(\sum_{n=1}^\infty \alpha_n e_n) = \sum_{n=1}^k \alpha_n e_n,$$
$$\sum_{n=1}^\infty \alpha_n e_n \in E.$$  

Then, an operator $T \in L(F, E)$ is compact if and only if $P_k T$ converges to $T$ in $L(F, E)$.

Proof:
The sufficiency part follows from the fact that for every Banach space $E$ and every normed space $F$, the limit in $L(F, E)$ of a sequence of finite-rank operators is a compact operator (Hirsch and Lacombe, 1999).

Now, suppose that $T \in L(F, E)$ is compact. For each positive integer $k$, let $T_k = P_k T$. Note that since
is orthonormal, it follows by Theorem 1.1 that $P_k \in L(F, E)$ and $\|P_k\| = 1$ for all $k$. Clearly we have, since $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of $E$,

$$\lim_{k \to \infty} P_k(y) = y, \text{ for each } y \in E$$

Let $B$ be the closed unit ball in $F$. Since $T$ is compact, it follows that $K = \text{cl}(T(B))$ is a compact subset of $E$. We need to show that

$$\limsup_{k \to \infty} \|T_k(x) - T(x)\| = 0.$$  

Suppose this is not true. Then there exist $\varepsilon > 0$, a subsequence $\{T_{k_j}\}$, and a sequence $\{x_{k_j}\}$ in $B$ such that

$$\|T_{k_j}(x_{k_j}) - T(x_{k_j})\| \geq \varepsilon, \text{ for all } j \quad (4)$$

Since $K$ is compact, there exists a subsequence of $\{x_{k_j}\}$, say $\{x_{j}\}$, such that the sequence $\{T(x_{j})\}$ converges in $K$ to some $y \in K$. Then we have, since $\|P_{k_j}\| = 1$ for all $j$,

$$\|T_{k_j}(x_{j}) - T(x_{j})\| \leq \|P_{k_j}(T(x_{j})) - P_{k_j}(y)\| + \|T(x_{j}) - P_{k_j}(y)\|$$

$$\leq \|T(x_{j}) - y\| + \|T(x_{j}) - P_{k_j}(y)\|$$

Letting $j \to \infty$, since $\{T(x_{j})\}$ and $\{P_{k_j}(y)\}$ both converge to $y$, we obtain that

$$\lim_{j \to \infty} \|T_{k_j}(x_{j}) - T(x_{j})\| = 0,$$  

which contradicts Re. (4). As a corollary we have

**Corolary 2.2:**

If $E$ is a Banach space that admits an orthonormal Schauder basis and $F$ is a normed space, then an operator $T \in L(F, E)$ is compact if and only if it is the limit in $L(F, E)$ of a sequence of finite-rank operators.

**Conclusion:**

In this work, we extend the usual notion of orthogonality to Banach spaces. Also, we establish a characterization of compact operators on Banach spaces that admit orthonormal Schauder bases. In (Hirsch and Lacombe, 1999), it is proved that every compact operator on Hilbert spaces is a limit of a sequences of finite-rank operators. In this paper, we extended this famous theorem, on Banach spaces. It is a guess of authors that this theorem can be extended on topological vector spaces. It is an open problem, now.

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**REFERENCES**


