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# Fuzzy Distance Measure for Fuzzy Numbers

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**Abstract:** In this paper a formula of fuzzy distance measure for fuzzy numbers is presented and also the metric properties of the proposed method are studied. Two numerical examples are illustrated for applying the method and the results of the modified method are compared with the previous methods.

**Key words**: Fuzzy distance, Generalized fuzzy number, LR-type fuzzy number, Interval arithmetic, Metric, Ambiguity, Fuzziness.

## INTRODUCTION

The distance between fuzzy numbers is a topic that has been studied by many researcher (Diamond, 1988; Voxman, 1998; Xu, 2001; Yang, 1997). Voxman first introduced a fuzzy distance for genralized fuzzy numbers (GFN) using the concept of α-cut. We note that other ways for defining distances between fuzzy sets have been described in the literature; see e.g., (Schweizer, 1983).

In a paper by C. Chakraborty and D. Chakraborty fuzzy distance measure (FDM) was suggested. In real applications, we find that the FDM formula for fuzzy numbers provided in (Chakraborty, 2006) is incorrect and may led to some misapplication. To avoid possible more misapplication or spread in the future, we present in this paper the correct FDM formula for fuzzy numbers and justify them theoretically.

Consider two trapezoidal fuzzy numbers $A$ = (2,4,1,2) and $B$ = (1,3,1,3). By FDM of (Chakraborty, 2006) with λ = 1, $d(A,B)=(-1,3,-1,\frac{3}{2})$ will be obtained. Also if $A$ = (1,3,1,2) and $B$ = (1,3,1,3) with λ = 0 the invalid result $d$ ($A,B$) = (-1,3,-1,2) will be obtained which has negative value as its left spread. To overcome the above-mentioned problems, we propose modified FDM in Section 3. The rest of the paper is organized as follows. Section 2 describes the basic notation and definitions of fuzzy arithmetic and interval arithmetic. Section 3 presents the correct FDM formula for GFNs and LR-type fuzzy number. The metric properties for correct FDM formula are studied in Subsection 3.3. Section 4 illustrates with some numerical example the fact that the incorrect formula by C. Chakraborty and D. Chakraborty can significantly led to negative left spread of fuzzy distance. Finally, conclusions are drawn in Section 5.

### *Preliminaries:*

A fuzzy set $A$ on a set universal $X$ is defined by membership function such that $\mu_A : X \to [0,1]$. The support of $A$, *supp* ($A$) is the closure of the set $\{x \in X \mid \mu(x) > 0\}$ and for each α ε [0, 1] the α-cut of $A$ is defined by $\{x \in X \mid \mu(x) \geq 0\}$.

### *Gfn:*

A GFN $A$, represented by $A = (\alpha_1, \alpha_2, \beta, \gamma)$ i.e., (left point, right point, left spread, right spread), is a normalized convex fuzzy subset (Zimmermann, 2001) on the real line ℝ if

(i) *supp* ($A$) is a closed and bounded interval, i.e., $[\alpha_1-\beta, \alpha_2+\gamma,]$,

(ii) $\mu_A$ is an upper semi-continuous function,

(iii) $\alpha_1-\beta < \alpha_1 \leq \alpha_2 < \alpha_2+\gamma$, and

(iv) the membership function is of the following form:

$$\mu_A(x)=\begin{cases} f(x) & x\in[a_1-\beta,a_1] \\ 1 & x\in[a_1,a_2] \\ g(x) & x\in[a_2,a_2+\gamma] \end{cases}$$

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where $f(x)$ and $g(x)$ are the monotonic increasing and decreasing functions in $[\alpha_1-\beta, \alpha_1]$ and $\alpha_2, \alpha_2+\gamma$ respectively. Specifically an LR-type fuzzy number is obtained from a GFN if the shape functions $f(x)$ and $g(x)$ are approximated by $L(\frac{a_1-x}{\beta})$ and $R(\frac{x-a_2}{\gamma})$ respectively. In particular, $A$ is trapezoidal in shape if $\alpha_1 < \alpha_2$ and triangular if $\alpha_1 = \alpha_2$. The α -cut of $A$ is an interval number denoted by $[A]_\alpha = [A^L(\alpha), A^R(\alpha)]$, which is explicitly shown for an LR-type fuzzy number, i.e., $[A^L(\alpha), A^R(\alpha)] = [\alpha_1-\beta L^{-1}(\alpha), \alpha_2+\gamma R^{-1}(\alpha)]$ for all α ε [0,1].

***Interval Arithmetic:***

Let us consider two interval numbers $[\alpha,b]$ and $[c,d]$ where $\alpha \le b$ and $c \le d$. Then the following arithmetic operations (Zimmermann, 2001) proceed as shown below:

(i) Addition: $[a,b]+[c,d]=[a+c,b+d]$ ,

(ii) Subtraction: $[a,b]-[c,d]=[a-d,b-c]$ ,

(iii) Multiplication: $[a,b]\times[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]$ ,

(iv) Division: $[a,b]\div[c,d]=[a,b]\times[\frac{1}{d},\frac{1}{c}]$ .

***Fuzzy distance given by Voxman:***

Here we briefly describe the fuzzy distance measure proposed by Voxman (Voxman, 1998). If the α-cut representations of $A = (\alpha_1, \alpha_2, \beta_1, \gamma_1)$ and $B = (b_1, b_2, \beta_2, \gamma_2)$ are $[A^L(\alpha), A^R(\alpha)]$ and $[B^L(\alpha), B^R(\alpha)]$ for all α ε [0,1]., respectively, then the α-cut representation of $d_{\text{voxman}}$ (Voxman, 1998), [L (α), R (α)] is given by

$$L(\alpha)=\begin{cases}\max\{B^L(\alpha)-A^R(\alpha),0\}, & \frac{A^L(1)+A^R(1)}{2}\le\frac{B^L(1)+B^R(1)}{2}\\ \max\{A^L(\alpha)-B^R(\alpha),0\}, & \frac{B^L(1)+B^R(1)}{2}\le\frac{A^L(1)+A^R(1)}{2}\end{cases}$$

and

$$R(\alpha)=\max\{A^R(\alpha)-B^L(\alpha),B^R(\alpha)-A^L(\alpha)\}.$$

***Modified FDM for Fuzzy Numbers:***

In this section, we take the point of view that the distance between two fuzzy numbers should itself be fuzzy. To improve FDM of (Chakraborty, 2006) we proposed a FDM for both case: (i) GFNs and (ii) LR-type fuzzy numbers.

***Defining Correct FDM for GFNs:***

Suppose $A = (\alpha_1, \alpha_2, \beta_1, \gamma_1)$ and $B = (b_1, b_2, \beta_2, \gamma_2)$ be two GFNs. Therefore the α-cut of $A$ and $B$ are $[A]_\alpha = [A^L(\alpha), A^R(\alpha)]$ and $[B]_\alpha = [B^L(\alpha), B^R(\alpha)]$ for all α ε [0,1], respectively. Now using of interval-difference operation for $[A]_\alpha$ and $[B]_\alpha$ for every α ε [0,1] can be one of the following:

$$for\ \lambda=\begin{cases}1, & \frac{B^L(1)+B^R(1)}{2}\le\frac{A^L(1)+A^R(1)}{2}\\ 0, & \frac{A^L(1)+A^R(1)}{2}\le\frac{B^L(1)+B^R(1)}{2}\end{cases} \tag{1}$$

where

$$d_\alpha^L=\max\{\lambda[A^L(\alpha)-B^L(\alpha)+A^R(\alpha)-B^R(\alpha)]+[B^L(\alpha)-A^R(\alpha)],0\}\ and$$

$$d_\alpha^R=\lambda[A^L(\alpha)-B^L(\alpha)+A^R(\alpha)-B^R(\alpha)]+[B^R(\alpha)-A^L(\alpha)].$$

Hence the fuzzy distance between $A$ and $B$ is defined by $d(A,B)=(d^L_{\alpha=1}, d^R_{\alpha=1}, \theta, \sigma)$ where;

$$\theta = d^L_{\alpha=1} - \int_0^1 d^L_\alpha \, d\alpha \quad and \quad \sigma = \int_0^1 d^R_\alpha \, d\alpha - d^R_{\alpha=1}. \tag{2}$$

***Defining Correct FDM for Lr-type Fuzzy Numbers:***

Let us consider LR-type fuzzy numbers, therefore the α-cut of $A$ and $B$ are as follows:

$$[A]_\alpha = [a_1 - \beta_1 L_1^{-1}(\alpha), a_2 + \gamma_1 R_1^{-1}(\alpha)] and [B]_\alpha = [b_1 - \beta_2 L_2^{-1}(\alpha), b_2 + \gamma_2 R_2^{-1}(\alpha)]$$

for all α ε = [0,1]. Hence the fuzzy distance between LR-type fuzzy numbers $A$ and $B$ obtained using Eq. (2) can be explicitly written as;

$$d(A,B) = (d^L_{\alpha=1}, d^R_{\alpha=1}, \theta, \sigma)$$

where

$$\theta = d^L_{\alpha=1} - \int_0^1 d^L_\alpha \, d\alpha \quad and \quad \sigma = \int_0^1 d^R_\alpha \, d\alpha - d^R_{\alpha=1}$$

such that

$$\begin{aligned} d^L_\alpha = \max\{&\lambda[(a_1+a_2)-(b_1+b_2)+\beta_2 L_2^{-1}(\alpha)-\beta_1 L_1^{-1}(\alpha)+\gamma_1 R_1^{-1}(\alpha)-\gamma_2 R_2^{-1}(\alpha)] \\ &+[(b_1-a_2)-(\beta_2 L_2^{-1}(\alpha)+\gamma_1 R_1^{-1}(\alpha))], 0\} \end{aligned}$$

$$\begin{aligned} d^R_\alpha = &\lambda[(a_1+a_2)-(b_1+b_2)+\beta_2 L_2^{-1}(\alpha)-\beta_1 L_1^{-1}(\alpha)+\gamma_1 R_1^{-1}(\alpha)-\gamma_2 R_2^{-1}(\alpha)] \\ &+[(b_2-a_1)+(\beta_1 L_1^{-1}(\alpha)+\gamma_2 R_2^{-1}(\alpha))] \end{aligned}$$

and

$$\begin{aligned} d^R_\alpha = &\lambda[(a_1+a_2)-(b_1+b_2)+\beta_2 L_2^{-1}(\alpha)-\beta_1 L_1^{-1}(\alpha)+\gamma_1 R_1^{-1}(\alpha)-\gamma_2 R_2^{-1}(\alpha)] \\ &+[(b_2-a_1)+(\beta_1 L_1^{-1}(\alpha)+\gamma_2 R_2^{-1}(\alpha))] \end{aligned}$$

***Metric Properties:***

(i) $d(A,B)=(d^L_{\alpha=1}, d^R_{\alpha=1}, \theta, \sigma)$ is a positive fuzzy number from Eq. (2),

(ii) $d\ (A,B) = d\ (B,A)$ by Eq. (1),

(iii) for $A$, $B$ and $C$, the proposed FDM satisfies the triangular property

$$d(A,C) \le d(A,B) \oplus d(B,C) \tag{3}$$

where $\oplus$ denotes the extended sum operator.

***Proof:***

i. Let $A = (\alpha_1, \alpha_2, \beta_1, \gamma_1)$ and $B = (b_1, b_2, \beta_2, \gamma_2)$ be two trapezoidal fuzzy numbers with the α-cut representation $[A]_\alpha = [\alpha_1 + (\alpha - 1)\beta_1, \alpha_2 + (1-\alpha)\gamma_1]$ and $[B]_\alpha = [b_1 + (\alpha - 1)\beta_2, b_2 + (1-\alpha)\gamma_2]$, respectively.

Now if $\dfrac{B^L(1)+B^R(1)}{2} \le \dfrac{A^L(1)+A^R(1)}{2}$ $(\lambda = 1)$ , we have

$$d^L_\alpha = \max\{(a_1-b_2)+(\alpha-1)(\beta_1+\gamma_2), 0\} \quad and \quad d^R_\alpha = (a_2-b_1)+(1-\alpha)(\beta_2+\gamma_1)$$

Hence three states exist.

1. if $d_\alpha^L = 0$ from Eq. (2) obtain $\theta = 0$ *and* $\sigma = \frac{\beta_2 + \gamma_1}{2}$,

then; $d(A,B) = (d_{\alpha=1}^L, d_{\alpha=1}^R, \theta, \sigma) = (0, d_{\alpha=1}^R, 0, \sigma) \geq 0$ .

2. if $d_\alpha^L = (a_1 - b_2) + (\alpha - 1)(\beta_1 + \gamma_2)$ , i.e., $(a_1 - b_2) - (\beta_1 + \gamma_2) \geq 0$ . we have $\theta = \frac{\beta_1 + \gamma_2}{2}$ *and* $\sigma = \frac{\beta_2 + \gamma_1}{2}$,

then $(a_1 - b_2) - \frac{\beta_1 + \gamma_2}{2} \geq 0$ therefore $d(A,B) \geq 0$.

3. if $d_\alpha^L = \begin{cases} 0 & 0 \leq \alpha \leq \omega \\ (a_1 - b_2) + (\alpha - 1)(\beta_1 + \gamma_2) & \omega < \alpha \leq 1 \ (a_1 - b_2 > 0) \end{cases}$

then $\theta = (\omega - 1)^2 \frac{\beta_1 + \gamma_2}{2} + \omega(a_1 - b_2) \geq 0$ *and* $\sigma = \frac{\beta_2 + \gamma_1}{2}$ hence $d\ (A,B) \geq 0$.

ii. The symmetry properties with respect to Subsection 3.1 and 3.2 is obvious.
iii. Let $A$ , $B$ and $C$ be there GFNs and their α-cut representation be expressed as $[A]_\alpha = [A^L(\alpha), A^R(\alpha)]$, $[B]_\alpha = [B^L(\alpha), B^R(\alpha)]$ and $[C]_\alpha = [C^L(\alpha), C^R(\alpha)]$ for all α ε [0,1]

Now if $\frac{A^L(1) + A^R(1)}{2} \leq \frac{B^L(1) + B^R(1)}{2} \leq \frac{C^L(1) + C^R(1)}{2}$, we have the fuzzy distance for

$A$ and $B$: $\begin{cases} d_\alpha^L = B^L(\alpha) - A^R(\alpha) \\ d_\alpha^R = B^R(\alpha) - A^L(\alpha) \end{cases}$ $B$ and $C$: $\begin{cases} \bar{d}_\alpha^L = C^L(\alpha) - B^R(\alpha) \\ \bar{d}_\alpha^R = C^R(\alpha) - B^L(\alpha) \end{cases}$ $C$ and $A$: $\begin{cases} \hat{d}_\alpha^L = C^L(\alpha) - A^R(\alpha) \\ \hat{d}_\alpha^R = C^R(\alpha) - A^L(\alpha) \end{cases}$

By the definition of an α-cut representation and the definition of ⊕, Eq. (3.4) is equivalent to the following equation for each α,

$$d_\alpha^L + d_\alpha^R + \bar{d}_\alpha^L + \bar{d}_\alpha^R \geq \hat{d}_\alpha^L + \hat{d}_\alpha^R. \tag{4}$$

Therefore the left hand side of Eq. (4) can be rewritten as

$$\begin{aligned} d_\alpha^L + d_\alpha^R + \bar{d}_\alpha^L + \bar{d}_\alpha^R &= \int_0^1 \max\{B^L(\alpha) - A^R(\alpha), 0\} d\alpha + \int_0^1 (B^R(\alpha) - A^L(\alpha)) d\alpha \\ &+ \int_0^1 \max\{C^L(\alpha) - B^R(\alpha), 0\} d\alpha + \int_0^1 (C^R(\alpha) - B^L(\alpha)) d\alpha \\ &= \int_0^1 \max\{B^L(\alpha) - A^R(\alpha), 0\} d\alpha + \int_0^1 \max\{C^L(\alpha) - B^R(\alpha), 0\} d\alpha \\ &+ \int_0^1 (C^R(\alpha) - A^L(\alpha)) d\alpha + \int_0^1 (B^R(\alpha) - B^L(\alpha)) d\alpha \qquad [\blacklozenge\ B^L(\alpha) \leq B^R(\alpha)] \\ &\geq \int_0^1 \max\{C^L(\alpha) - A^R(\alpha), 0\} d\alpha + \int_0^1 (C^R(\alpha) - A^L(\alpha)) d\alpha \\ &= \hat{d}_\alpha^L + \hat{d}_\alpha^R. \end{aligned}$$

***Ambiguity and Fuzziness Within a Fuzzy Number:***

The concepts of ambiguity and fuzziness were defined as follows (Delgado, 1998). Let $A$ be a fuzzy number with α-cut representation, $[A^L(\alpha), A^R(\alpha)]$ and $S$ be a reducing function. Then the ambiguity of $A$ is

given by

$$Amb(A) = \int_0^1 s(\alpha)[A^R(\alpha) - A^L(\alpha)]d\alpha, \tag{5}$$

if *Supp* (*A*) = [$\alpha_1 - \beta$, $\alpha_2 + \gamma$] , then the fuzziness of *A* is computed as *Fuzz* (*A*) where;

$$\begin{aligned} Fuzz(A) &= \int_0^1 s(\alpha)[a_2 + \gamma - a_1 + \beta]d\alpha \\ &-\{\int_{\frac{1}{2}}^1 s(\alpha)[A_c^L(\alpha) - a_1 + \beta]d\alpha + \int_{\frac{1}{2}}^1 s(\alpha)[A^R(\alpha) - A^L(\alpha)]d\alpha + \\ &\int_{\frac{1}{2}}^1 s(\alpha)[a_2 + \gamma - A_c^R(\alpha)]d\alpha + \int_0^{\frac{1}{2}} s(\alpha)[A_c^L(\alpha) - a_1 + \beta]d\alpha + \\ &\int_0^{\frac{1}{2}} s(\alpha)[A_c^R(\alpha) - A_c^L(\alpha)]d\alpha + \int_0^{\frac{1}{2}} s(\alpha)[a_2 + \gamma - A_c^R(\alpha)]d\alpha\}. \end{aligned} \tag{6}$$

Human intuition suggests that vagueness definitely exists in the distances between any two fuzzy numbers; but a less vague and ambiguous distance is always acceptable from the stability point of view. In order to compare the fuzzy distance measure obtained by the proposed approach with that given by Voxman's approach, fuzziness and ambiguity measures have been considered. Suppose we have two fuzzy numbers *A* and *B*, with the same central value (s) but with different spreads. Then *A* is expected to be better than *B* in the sense of stability or preciseness if

$$Amb(A) < Amb(B) \quad and \quad Fuzz(A) < Fuzz(B).$$

Considering this, two propositions are demonstrated here to compare between the proposed measure, $d_{proposed}$, and Voxman's measure, $d_{voxman}$ , as follows;

***Remark 3.1:***

For λ =0, 1, $L(\alpha)$ = 0 if and only if $d_\alpha^L = 0$

***Proposition 3.1:***

In the consideration of two GFNs *A* and *B* with their α -cuts [$A^L$ (α), $A^R$ (α)] and [$B^L$ (α), $B^R$ (α)], the ambiguity of the fuzzy distance obtained from the proposed measure is less than or equal to that of Voxman's measure, i.e., *Amb* ($d_{proposed}$)≤ *Amb* ($d_{voxman}$).

***Proof:***

The α-cut of $d_{voxman}$ [6] can be rewritten as follows:

$$L(\alpha) = \begin{cases} \max\{B^L(\alpha) - A^R(\alpha), 0\}, \ \lambda = 0 \\ \max\{A^L(\alpha) - B^R(\alpha), 0\}, \ \lambda = 1 \end{cases} \text{ and } R(\alpha) = \max\{A^R(\alpha) - B^L(\alpha), B^R(\alpha) - A^L(\alpha)\}.$$

***State 1:***

Suppose for $\lambda$ = 0,1, $L$ ($\alpha$) = 0.

If $A^R(\alpha) - B^L(\alpha) \geq B^R(\alpha) - A^L(\alpha)$ then we have

$$Amb(d_{Voxman}) = \int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha, \tag{7}$$

also if $A^R(\alpha) - B^L(\alpha) \leq B^R(\alpha) - A^L(\alpha)$ then

$$Amb(d_{Voxman}) = \int_0^1 s(\alpha)[B^R(\alpha) - A^L(\alpha)]d\alpha. \tag{8}$$

From remark (3.1) for λ =0, 1, have $d_\alpha^L = 0$ If λ = 0 we obtain.

$$Amb(d_{proposed}) = \int_0^1 s(\alpha)[B^R(\alpha) - A^L(\alpha)]d\alpha. \tag{9}$$

Eqs. (7-9) show *Amb* ($d_{\text{proposed}}$)≤ *Amb* ($d_{\text{voxman}}$). For λ = 1 represent such as.

***State 2:*** Suppose for λ = 0, 1, *L* (α)≠0. From remark (3.1) $d_\alpha^L \neq 0$, therefore obtain

$$Amb(d_{proposed}) = \int_0^1 s(\alpha)[A^R(\alpha) + B^R(\alpha) - A^L(\alpha) - B^L(\alpha)]d\alpha. \tag{10}$$

Now for λ = 0, 1,

$$Amb(d_{Voxman}) = \begin{cases} \int_0^1 s(\alpha)[A^R(\alpha) + B^R(\alpha) - A^L(\alpha) - B^L(\alpha)]d\alpha & (z) \\ 2\int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha & (zz) \\ 2\int_0^1 s(\alpha)[B^R(\alpha) - A^L(\alpha)]d\alpha & (zzz) \end{cases} \tag{11}$$

i. Eqs. (10) and (11z) show *Amb* ($d_{\text{proposed}}$)≤ *Amb* ($d_{\text{voxman}}$).

ii. For comparing Eq. (10) with Eq. (11zz), we have $B^R(\alpha) - A^L(\alpha) \leq A^R(\alpha) - B^L(\alpha) < 0$, there fore

$$\begin{aligned} &= Amb(d_{Voxman}) \\ &= 2\int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha \\ &= \int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha + \int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha \\ &> \int_0^1 s(\alpha)[A^R(\alpha) - B^L(\alpha)]d\alpha + \int_0^1 s(\alpha)[B^R(\alpha) - A^L(\alpha)]d\alpha \\ &= \int_0^1 s(\alpha)[A^R(\alpha) + B^R(\alpha) - A^L(\alpha) - B^L(\alpha)]d\alpha \\ &= Amb(d_{proposed}) \end{aligned}$$

iii) such as (ii) compare Eq. (10) with Eq. (11zzz).

***Proposition 3.2:***

With the α-cuts [$A^L$ (α) , $A^R$ (α)] and [$B^L$ (α) , $B^R$ (α)] of two GFNs *A* and *B*, the fuzziness of the fuzzy distance obtained from the proposed measure is less than or equal to that for Voxman's measure, i.e., *Amb* ($d_{\text{proposed}}$)≤ *Amb* ($d_{\text{voxman}}$).

***Proof:***

see (Chakraborty, 2006).

***Numerical Examples:***

***Example 4.1:***

Let $A$ = (2,4,1,2) and $B$ = (1,3,1,3) be two trapezoidal fuzzy numbers. Then their α-cut representations are as follows

$$[A]_\alpha = [\alpha+1, 6-2\alpha] and [B]_\alpha = [\alpha, 6-3\alpha],$$

with respect to (3.1) of (Chakraborty, 2006), λ = 1. Therefore due to formula of (Chakraborty, 2006).

$$d_\alpha^L = 4\alpha - 5 \quad and \quad d_\alpha^R = 6-3\alpha,$$

$$d(A,B) = (d_{\alpha=1}^L, d_{\alpha=1}^R, \theta, \sigma) = (-1, 3, \theta, \sigma),$$

and

$$\theta = d_{\alpha=1}^L - \max\{\int_0^1 d_\alpha^L d\alpha, 0\} = -1 - \max\{\int_0^1 (4\alpha-5) d\alpha, 0\} = -1,$$

$$\sigma = \int_0^1 d_\alpha^R d\alpha - d_{\alpha=1}^R = \int_0^1 (6-3\alpha) d\alpha - 3 = \frac{3}{2}.$$

It is clear that the left spread of fuzzy distance is negative, such a result is unacceptable. So fuzzy distance formula of (Chakraborty, 2006) is incorrect. However, by our correct formula of Subsection (3.1).

$$d_\alpha^L = 0, \quad d_\alpha^R = 6-3\alpha, \quad \theta = 0, \quad \sigma = \frac{3}{2}$$

and the fuzzy distance between $A$ and $B$ is $d(A,B) = (0,3,0,\frac{3}{2})$. Now, by Voxman formula (Voxman, 1998) $d$ $(A,B)$ = (0,3,0,3). Also with $s$ (α) = α, the ambiguity and fuzziness of $d_{\text{proposed}}$ and $d_{\text{voxman}}$ are computed using Eqs. (5) and (6) as $Amb(d_{proposed}) = \frac{7}{4}, Amb(d_{Voxman}) = 2; Fuzz(d_{proposed}) = \frac{3}{8}, Fuzz(d_{Voxman}) = \frac{3}{4}$

***Example 4.2:***

Let $A$ = (-1,2,1,3) and $B$ = (0,3,1,5) be two LR-type fuzzy numbers with shape functions

$$L(x) = \frac{1}{1+5x}, R(x) = \frac{1}{1+2x}.$$

Then their α-cut representations are as follows;

$$[A]_\alpha = [-1-\frac{1-\alpha}{5\alpha}, 2+\frac{3(1-\alpha)}{2\alpha}] and [B]_\alpha = [\frac{3(1-\alpha)}{5\alpha}, 1+\frac{5(1-\alpha)}{2\alpha}].$$

With respect to (3.1) of [1], λ = 1 and

$$d_\alpha^L = -2 - \frac{27(1-\alpha)}{10} and d_\alpha^R = 2 + \frac{21(1-\alpha)}{10},$$

then

$$d(A,B) = (d_{a=1}^L, d_{a=1}^R, \theta, \sigma) = (-2, 2, \theta, \sigma),$$

and

$\theta = -2, \sigma = \frac{21}{10}.$

It is clear that the left spread of fuzzy distance is negative, such a result is unacceptable. So fuzzy distance formula of (Chakraborty, 2006) is incorrect. However, by our correct formula of Subsection (3.1) fuzzy distance between $A$ and $B$ is $d(A,B) = (0,2,0,\frac{21}{20})$ and by Voxman formula (Voxman, 1998), $d(A,B) = (0,2,0,\frac{21}{10})$. There fore, with s(α)= α the ambiguity and fuzziness of $d_{\text{proposed}}$ and $d_{\text{voxman}}$ are computed using Eqs. (5) and (6) as;

$$.Amb(d_{proposed}) = \frac{47}{40}, Amb(d_{Voxman}) = \frac{27}{20}; Fuzz(d_{proposed}) = \frac{21}{80}, Fuzz(d_{Voxman}) = \frac{21}{40}$$

***Conclusions:***

In this paper, the FDM formula for fuzzy numbers provided by (Chakraborty, 2006) were shown to be incorrect. To avoid possible further misapplication or spread in the future, we presented the modified FDM formula for fuzzy numbers. We also justified the presented formula by proving the properties. Two numerical examples demonstrated that incorrect formula could significantly led to negative left spread for fuzzy distance and the correct formula overcomes that problem.

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