Review on Solution Methods for Random Vibration Problems

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Abstract: Random vibration is becoming more and more important for many applications. It poses a serious challenge to the researchers because of its complex nature. In this paper we tried to classify the existing random vibration solution techniques and explain them in clear manner that will help any researcher in the future to apply the right approach for the right problem.

Key words: random vibration; linear systems; nonlinear systems, parametrically excited

INTRODUCTION

In dynamic systems we face different types of vibrations. In general, we classify them broadly as regular vibration which can be described easily with a mathematical form and random vibration which does not lend itself for a mathematical form. The first type has a lot of solution methodologies that can be applied to different systems. Random vibrations do lend itself easily to solutions. Most of engineering applications are in fact randomly oscillated, this calls for the need of having a handy, clear review of the current status in random vibration techniques. This study owes much to many extensive reviews on random vibration techniques such as Roberts and Spanos (1990), Crandall (1983) Roberts (1984,1988,1990), and Spanos (1989).

Solution Methods:

Dynamic systems are classified mainly into three categories: linear time-invariant systems, nonlinear systems, and parametrically excited systems.

I. Linear Time-invariant Systems:

This kind of systems is represented mathematically as:

\[ M\ddot{X} + C\dot{X} + KX = F(t) \]  

Where
- \( M \): system mass
- \( C \): system damping
- \( K \): system stiffness
- \( F(t) \): input excitation

It should be noted that \( M, C \) and \( K \) are independent of time. For a single input- single output system, if the system frequency response function is \( H(f) \), and the input is stationary the output response will be stationary also. This is could be presented as:

\[ M\pi = H(0) Me \]  

\[ W\pi(f) = H(f) We(f) \]

Where,
- \( M\pi \): response mean value
- \( Me \): excitation mean value

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Such systems could be solved very easily by any available numerical method for ordinary differential equations. The general properties of such systems are:

a) Their response is independent of the probability distribution involved i.e., the probability distribution of the output is the same as the input, which characterizes any linear system.

b) Stationarity is maintained between input and output unless the system is undamped or unstable in which cases the output could be sometimes nonstationary.

For the case of nonstationary inputs, the input-output relations have been given by Mark’s instantaneous spectrum. Recently, the stochastic averaging method, which will be explained later, has been applied to obtain approximate statistics when the excitation is modeled as an evolutionary process or as a modulated stationary process.

II. Nonlinear Systems:

Most of the engineering problems are nonlinear. Therefore, a number of effective methods were devised to solve non-linear stochastic systems. In the case of a single input-output system, we are faced by the problem of solving a nonlinear stochastic differential equation, but when we have a multi-input-output system, a set of coupled stochastic ordinary differential equations exists. Therefore, it is necessary to specify not only the statistical characteristics of each individual input or output process, but also the statistical relationship between the various processes.

1. Statistical Linearization:

In statistical linearization method the governing set of nonlinear differential equations are replaced by an equivalent set of linear equations, and the difference between the sets being minimized in some appropriate sense. Therefore, if we have the following general form of stochastic differential equation,

\[ g(Y) = X(t) \]  

Where,

\[ g(y) \]: arbitrary nonlinear function of \( Y \)

\[ X(t) \]

And \( Y(t) \) is represented as

\[ Y = [Y, \dot{Y}, \ddot{Y}]^T \]  

Where \( Y(t) \), \( \dot{Y}(t) \) and \( \ddot{Y}(t) \) are the response displacement, velocity and acceleration respectively.

Therefore, replacing equ. (1), with the equivalent linear form

\[ M\ddot{Y} + C\dot{Y} + KY = X(t) \]  

Where \( M \), \( C \), \( K \): are time-dependent parameters, if the statistical nature of \( X(t) \) varies with time.

Therefore, the error between equ. (4) and equ. (6) is

\[ e = g(Y) - [M\ddot{Y} - C\dot{Y} - KY] \]

which will be minimized in any convenient way.

The assumptions and limitations which stand behind the method are:

a) The response is assumed to be Gaussian distributed.

b) It does not reveal the effect of nonlinearities on response distribution as we must assume it Gaussian.

c) It yields only estimates of the first and second moments of the response, which are not sufficient for some applications.
d) It does not give any information about system stability. Therefore it is not applied to parametric or multiplicative excitation.

2. Moment Closure:
In this method, the equations for the moments of response such as the mean, mean square and mean cube can be derived readily from the equations of motion.

So, from these moments or related quantities it is possible to derive estimates of the probability distribution of the response using a variety of analytical expansions or the Gaussian distribution. When dealing with nonlinear systems a main difficulty arises with this approach which is that the moments are generally governed by an infinite hierarchy of coupled equations, so we must introduce a closure approximation to obtain a soluble set of equations. The simplest level of closure is to assume that the response is Gaussian. In that case the final solution will be identical to that obtained from statistical linearization approach.

Crandall (1980) has introduced a non-Gaussian approach and Noori et al (1987) have used this approach to solve a nonlinear oscillator with single degree of freedom and with a restoring force in the form of hyperbolic tangent function. This solution was compared with another solution using statistical linearization technique which proved to be more accurate and efficient in comparison with the moment closure technique.

Finally, a characteristic feature of moment closure approach is that the complexity of the moment equations dramatically increases as the order of closure increases.

3. Equivalent Nonlinear Equation:
It is a kind of generalization of statistical linearization approach. The idea is to replace the original set of nonlinear differential equations by an equivalent set of exactly solvable nonlinear equations. This approach is still very limited in application.

Unlike the statistical linearization this method gives us information about the departure of the response distribution from the Gaussian one.

4. Perturbation and Functional Series:
Both methods are of expansion type, and they are applied in case of sufficiently small nonlinearities in the system.

In perturbation technique, the basic idea is to expand the solution to a nonlinear set of equations in terms of small scaling parameter, \( \lambda \) say which characterizes the magnitude of the nonlinear terms. The first term of the expansion is the linear response when all nonlinearities are vanished; subsequent terms express the influence of nonlinearity. In practice, due to the lengthy calculations and tedious work as the order of \( \lambda \) increase, results are usually obtained only up to the first order of \( \lambda \). Functional series methods offer an alternative approach to develop an expansion based on the linear solution. A common difficulty with all expansion methods lies in establishing the regimes of convergence in the appropriate parameter space.

5. Fokker – Plank – Kolmogorov (FPK) Equation:
An entirely different approach is used based on the theory of Markov continuous processes which are also called diffusion processes. The Markov or diffusion is the continuous random process \( X(t) \) in which the future behavior at time \( t_{n+1} \), provided its present value \( X(t_n) \) is given, does not change by additional information of its past. This property led some people to name the Markov process as “one step memory stochastic process” or “process without after effect”.

The Markov process is a little bit more realistic stochastic process than the white noise process which is a purely mathematical process completely random, totally unpredictable and not correlated with its earlier values.

Continuous Markov processes are described in terms of the conditional probability density

\[
P(X_{n+1}, X_n, X_{n-1}, \ldots, X_0) = P(X_{n+1}/X_n) \]  

(8)

The form of conditional probability of Markov processes is called the transition probability since

\( P(X_{n+1}/X_n) \) gives the density of probability of a transition from one point in the state space \( X \) at time
to a point in the phase space $X_n$ at time $t_{n+1}$.

The relation that governs the transition probability density of a Markov process is an integral equation of the form

$$P(X_{n+1} \mid X_n) = \int_{-\infty}^{\infty} P(X_{n+1} \mid X_n) P(X_n \mid X_{n-1}) dX_n$$

(9)

This equation is called the forward Smoluchowski Chapman Kolmogorov equation. The solution of the Smoluchowski equation is usually obtained by solving a partial differential equation established independently by Fokker, Planck and Kolmogorov. The $FPK$ equation has two forms mainly: the forward and backward equations. The derivation and the forms of the equation are presented in appendix (A) in a simplified way.

When the excitation is a white noise process, the response is Markov process. The transition probability density function, which together with an initial distribution completely describes the response, satisfies the $FPK$ equation. The variable coefficients of the $FPK$ equation can be derived from the nonlinear equations of motion of the dynamic system. Two procedures have been developed for this:

a) Drift and diffusion coefficients are obtained by limit operations applied directly to the equations of motion.

b) The equations of motion are replaced by an equivalent set of Ito- equations according to certain translation rules and the drift and diffusion coefficient obtained by formal operations on the coefficients of Ito-equations.

Exact solutions of $FPK$ equation can be found only in special cases which usually relate the steady-state response to stationary excitation.

For second order systems driven by white noise, a general form of the equation of motion is,

$$\ddot{X} + g(X, \dot{X}) = \xi(t)$$

(10)

Where,

$g$ : is a function of displacement $X$ and velocity $\dot{X}$

$\xi(t)$ : is a white noise process

Exact solution exists only for some specific forms of $g$.

Approximate numerical solutions for $FPK$ are available, the most popular are:

a) Series solution to $FPK$ equations in terms of eigenfunctions, but in only a few cases can these be found analytically.

b) Stochastic averaging which is applied for lightly-damped nonlinear oscillators responding to wide-band excitation the response is narrow band in character. To apply $FPK$ equation to dynamic systems, the following conditions should be satisfied:

a) The two forms of the $FPK$ equation can only be used if the random excitation is Gaussian white noise or filtered white noise.

b) The response of the dynamic systems should be approximated as a Markov process to be handled.

c) The dynamic system must have an equation of motion to represent the system.

d) The equations of motion should not include nonlinear coupling of damping or inertia.

e) If damping exists in the dynamic system, the damping forces should be proportional to the velocities. Also, the correlation function matrix of the excitation should be proportional to the system damping matrix.

6. Monte Carlo Simulation Technique:

This method is one of the most powerful tools to solve nonlinear random systems. The theoretical foundation of Monte Carlo formulation is associated with the fact that the stochastic differential equations governing the motion of the system can be interpreted an infinite set of deterministic differential equations.

Simulation techniques offer a practical approach when dealing with very difficult nonlinear problems and they are also a useful way to check the validity of approximate theoretical methods. In general a great number of samples, $n$, must be used especially for nonstationary cases, but this condition may be relaxed in case of stationary processes. Also, it should be noted that, in case of stationarity, under certain conditions, (Scheurkogel & Elishakoff 1985) ergodicity may be assumed.

The statistical uncertainty in the response statistics decreases in proportion to the square root of $(1/n)$ while the cost increases essentially in proportion to $n$, where $n$ is the number of trials. Therefore, to gain one additional figure in the result requires a hundredfold increase in cost.
Spanos (1981) estimated that for cases where statistical linearization and simulation are both applicable the former is much efficient of the latter.

III. Parametric Excitation:
A vibratory system in which the effective stiffness and or damping parameters are forced to vary with time is said to be parametrically excited.
For studying random parametric excitation two principal procedures have been developed namely, moments methods and methods based on Lyaponov function.

1. Moment Methods:
In which a set of ordinary differential equations is obtained for certain response moments and the system stability could be decided on the basis of the stability of the moments.

2. Methods Based on Lyaponov Function:
In which use is made of an energy-like non-negative function \( V \) of the response and the system stability decided on the basis of the sign of \( \frac{d}{dt} <V> \).
In implementing the above methods, which concern mainly of the stability of the system which is the most important criteria in studying parametric systems, it is often necessary to incorporate one or more of the approximate procedures on nonlinear methods e.g. linearization, averaging, moment closure etc.

Conclusions:
We tried to classify the solution techniques related to random vibrations in relevance to the type of dynamic system in hand. This will give a handy review for all researchers working randomly vibrated systems.

REFERENCES