Random Vibrations of a Beam on Elastic Foundation

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Abstract: Beam on an elastic foundation represents many applications like bridges, rail tracks. The analysis of such situation under dynamic loads is of quite importance to industry and researchers as well. Beam on elastic foundation represent a contact problem of the receding type. In this paper, we used a quasi-linear approach integrated with the Finite Element Method to model the contact problem. In this approach we exploit the standard form of the FEM for linear structures to solve a nonlinear problem instead of going into the nonlinear formulation. Then, we generate the random dynamic load using Monte-Carlo simulation. We study two different cases of beams on elastic foundation. We plot the displacement and velocity as outputs for this problem and analyze the input-output relationship from the randomness point of view.

Key words:

INTRODUCTION

Beam on elastic foundation is one of the classic problems in theory of elasticity and it went through a lot of different solution approaches (El-Meliegy, 1988). The interest in this problem is due to the wide applications that it has in engineering such as foundation building and railway rails. Most of the classic interest of this problem was focused on solving the static case and then the case under regular dynamic loads (Chen et al., 2004). In cases like railway rails the loads are more of a random nature. In this paper we are using the Finite Element Method to solve this problem under random loads assuming Winkler foundation.

Computational model:

There are many algorithms for contact-problems and none of them included random loading analysis. The main objectives behind choosing a contact algorithm were that the algorithm is tested for other cases successfully and that the algorithm lends itself to the ordinary linear finite element frame of work. These features were pronounced in the algorithm developed by Mahmoud et al., (1990).

The algorithm has been developed for elasto-static contact and tested for several numerical experiments showing a good agreement with other experimental, analytical and numerical techniques Mahmoud et al (1982, 1983, 1985, and 1986). The algorithm is then developed to handle contact-impact problems and tested for some typical examples e.g. beams and plates on elastic foundations, showing a good results Hassan (1988).

The mathematical model of the dynamic system of two masses in contact can be represented as follows:

\[
[M][\ddot{U}(t)] + [K]
\]

In \( \Omega_1 \) and \( \Omega_2 \) subjected to the boundary conditions

\[
\{U(t)\} = \{\bar{U}(t)\}
\]
on \( \Gamma_{s1} \) and \( \Gamma_{s2} \)
and the initial conditions

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\[ \{U(t)\} = \{U(t_0)\} \] (8a)
in \( \Omega_1, \Omega_2, \Gamma_1 \) and \( \Gamma_2 \)
\[ \{\dot{U}(t)\} = \{\dot{U}(t_0)\} \] (8b)
In \( \Omega_1, \Omega_2, \Gamma_1 \) and \( \Gamma_2 \)
Where \( \dot{U} \) is the velocity; \( \Gamma_1 \) is the boundary of the domain \( \Omega_1 \) and \( \Gamma_2 \) is the boundary of the domain \( \Omega_2 \), \( \Gamma_{vi} \) and \( \Gamma_{v2} \) are the parts of the boundaries where displacements are prescribed; \( \{U(t_0)\} \) and \( \{\dot{U}(t_0)\} \) denote prescribed initial displacement and velocities; and \( \{U(t)\} \) are the given surface displacements.

External load is dynamically applied over \( \Gamma_{vi} \) and \( \Gamma_{v2} \) attains the value \( F(t) \) at every step \( k \). In addition to the above mentioned boundary parts, boundaries \( \Gamma_{v1} \) and \( \Gamma_{v2} \) on which the unilateral boundary conditions are set. The relations in \( \Gamma_{v} \) at step \( k \) are given by

\[ U_{in}^k - U_{jn}^k - G_{ij} < 0 \text{ on inactive } \Gamma_{v1} \text{ and } \Gamma_{v2} \] (9a)

\[ U_{in}^k - U_{jn}^k - G_{ij} \leq 0 \text{ on active } \Gamma_{v1} \text{ and } \Gamma_{v2} \] (9b)

Where \( G_{ij} \) is the gap of the candidate pairs \( i \) and \( j \) and \( n \) denotes a direction from node \( i \) to node \( j \) which is approximately normal to the two surfaces.

Once the two bodies are discretized into finite elements and boundary and initial state conditions prescribed, they must be assembled. Referring to equ. (6), \([M]\) is the structure mass matrix

\[ [M] = \sum [M_e] \quad e = 1, 2, ..., nn, \] (10)

Where \( nn \) is the number of elements and \([M_e]\) is the element mass matrix

\[ [M] = \int p[N]^T[N]dv, \] (11)

Where \([N]\) is matrix of shape functions and \( p \) is the mass density of the element. The structure stiffness matrix is given by:

\[ [k] = \sum [k_e] \quad e = 1, 2, ..., nn, \] (12)

Where \([k_e]\) is the element stiffness matrix calculated from

\[ [k_e] = \int [B]^T[E][B]dv, \] (13)

Where \([E]\) is the matrix of elastic stiffness and \([B]\) is the nodal displacement-strain matrix given by

\[ [B] = \partial[N] \] (14)

Using equ (11) through (14), the equations of motion of the two bodies are given by:

\[ \begin{bmatrix} k_{d1} & k_{d2} \\ k_{v1} & k_{v2} \end{bmatrix} \begin{bmatrix} u_{d1} \\ u_{v1} \end{bmatrix} + \begin{bmatrix} m_{d1} & m_{d2} \\ m_{v1} & m_{v2} \end{bmatrix} \begin{bmatrix} \ddot{u}_{d1} \\ \ddot{u}_{v1} \end{bmatrix} = \begin{bmatrix} F_{d1} \\ F_{v1} + R_1 \end{bmatrix} \] (15)
\begin{align}
\begin{bmatrix}
k_{d1} & k_{d2} \\
n_{c1} & n_{c2}
\end{bmatrix}
\begin{bmatrix}
u_{d1} \\
u_{c1}
\end{bmatrix}
+ 
\begin{bmatrix}
m_{d1} & m_{d2} \\
m_{c1} & m_{c2}
\end{bmatrix}
\begin{bmatrix}
u_{d2} \\
u_{c2}
\end{bmatrix}
= 
\begin{bmatrix}
F_{d2} \\
F_{c2}
\end{bmatrix}
+ R_{1}
\end{align}

(16)

where \{u_{d1}\} and \{u_{c1}\} are the vectors of degrees of freedom on \(\Gamma_{d1}, \Gamma_{c1}\), \{u_{d2}\} and \{u_{c2}\} are the vectors of degree of freedom for the body except \{u_{c1}\}, respectively and \(R_{1}\) and \(R_{2}\) indicate the contact forces between the two bodies at the contact zones \(\Gamma_{d1}, \Gamma_{c2}\), respectively.

It is clear that in case of contact between two candidate nodes, the contact forces at that pair acting on the two bodies are equal in magnitude and act in opposite directions.

In the case of separation these forces are equal to zero. Adding equations (15) and (16), the equation of motion of the two bodies in the case of separation is given by:

\begin{align}
\begin{bmatrix}
k_{d1} & k_{d2} & 0 & 0 \\
n_{c1} & n_{c2} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{d1} \\
u_{c1}
\end{bmatrix}
+ 
\begin{bmatrix}
m_{d1} & m_{d2} & 0 & 0 \\
m_{c1} & m_{c2} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{d2} \\
u_{c2}
\end{bmatrix}
= 
\begin{bmatrix}
F_{d1} \\
F_{c1}
\end{bmatrix}
+ F_{d2}
\end{align}

(17)

Contacts are detected through calculation of the interpenetration \(C\) between deformable bodies

\(C_{ij}^k = u_{in}^k - u_{jn}^k - G_{ij}^k\)

on inactive \(\Gamma_{c}\)

where the sub-and superscripts are as defined previously.

If \(C\) is chosen such that

\(C_{ij} = MAX(C_{ij})\)

on inactive \(\Gamma_{c}\)

Then \(i\) and \(j\) are the succeeding contact points.

Separations are detected through calculation of the factor \(S\) given by

\(S_{ij}^k = -C_{ij}^k - (u_{in}^k - u_{jn}^k)\)

on active \(\Gamma_{c}\)

If \(S\) is chosen such that

\(S_{ij} = MAX(S_{ij})\)

on active \(\Gamma_{c}\)

Then \(i\) and \(j\) are the succeeding release points.

As far as we know no attempt has been taken to develop a simulation scheme that could handle randomly excited contact systems of several degrees of freedom. In this work we propose a general, easy to understand simulation scheme into which any incremental solution technique for dynamic contact systems, of any number of degrees of freedom, could be applied.
The scheme of the algorithm is as follows:

1. Input the preliminary data, which describe the onset of the problem (e.g. initial conditions, boundary conditions, system characteristics… etc).

2. Setting the equations in its incremental form

3. Solving the system of equations by one of the direct integral methods

4. Checking the contact or separation condition by an appropriate criterion i.e. according to displacement or force.

5. Correcting the system of equations according to the new contact conditions.

6. Changing the forcing function value according to a chosen random number generator, this is in our case Monte Carlo approach.

7. Going to step (3) through (6) again, until the chosen time domain of the problem satisfied.

8. Stop

Examples and Results:

The random load is assumed to be normal, stationary, and ergodic (Crandall, 1965).

As mentioned above, these assumptions are restrictive but they are commonly used as an acceptable assessment to engineering problems subjected to random vibrations tackled by numerical methods because of the huge computational effort that would be needed if a much more realistic situation is approached (Crandall & Zhu 1983).

The foundation models are quite many, but we will use the most simple and the most common of them which is the elastic Winkler foundation model which is represented simply by a series of equally spaced springs of linear stiffness.

The beam is discretised into 10 beam elements and resting on the spring elements representing the foundation. The force is applied at the middle node, so a half model will be sufficient to the analysis, which is represented in Fig. (1.1). the gaps are all initially zero and the time step=0.00001 sec. with time span = 0.12 sec.

Displacement and velocity response of the mid-node is shown in Fig. (1.2) and Fig. (1.3). It is significant that the randomness of the outputs is less than that noticed in the previous sample.

Fig. 1.2: Discretization of a beam an elastic foundation.

Fig. 1.2: Displacement of the midpoint of a beam on elastic foundation subjected to random excitation.
Conclusions:
In this paper we introduced a new computational scheme to calculate random loaded beam on elastic foundation based on the theoretical model proposed by Mahmoud et al (1982). We generated the random load using Monte Carlo Simulation. Although the input is strictly normal, the outputs show a clear deviation of normality, which is expected in such severe nonlinear problems. Also, although the input is stationary and ergodic, the outputs are clearly nonstationary and as a result of non-ergodic.

REFERENCES