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Coupled Fixed Point Theorems in Partially Ordered Metric Spaces Which Endowed With Vector-valued Metrics

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Abstract: In this paper, the existence and uniqueness of coupled fixed point for mapping having the mixed monotone property in partially ordered metric spaces which endowed with vector-valued metrics are given.

Key words:

INTRODUCTION

Let X be a nonempty set. A mapping $d: X \times X \to \mathbb{R}^m$ is called a vector-valued metric on X if the following properties are satisfied:

1. $d(x,y) \geq 0$ for each $x, y \in X$; if $d(x,y) = 0$, then $x = y$;
2. $d(x,y) = d(y,x)$ for each $x, y \in X$;
3. $d(x,y) \leq d(x,z) + d(z,y)$ for each $x, y, z \in X$.

A set X equipped with a vector-valued metric d is called a generalized metric space and denoted by (X,d). By $M_{m,m}(\mathbb{R}^+)$ we mean that the set of all $m \times m$ matrices with positive elements. We denote by Θ the zero matrix, and by I the identity $m \times m$ matrix. Let $A \in M_{m,m}(\mathbb{R}^+)$, A is said to be convergent to zero if and only if $A^n \to 0$ as $n \to \infty$ (for more details see (R.S. Varga, 2000).

Let $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, and $c \in \mathbb{R}$. By $\alpha \leq \beta$ (resp. $\alpha < \beta$) we mean that $\alpha_i \leq \beta_i$ (resp. $\alpha_i < \beta_i$) for each $1 \leq i \leq m$, and by $\alpha \leq c$ (resp. $\alpha < c$) for $1 \leq i \leq m$.

Notice that for the proof of the main results, we need the following equivalent statements

1. A is convergent towards zero;
2. $A^n \to 0$ as $n \to \infty$;
3. The eigenvalues of A are in the open uint disc, that is, $|\lambda| < 1$, for each $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$;
4. The matrix $I - A$ is nonsingular and $(I - A)^{-1} = I + A + \cdots + A^n + \cdots$;
5. $A^n q \to 0$ and $qA^n \to 0$ as $n \to \infty$, for each $q \in \mathbb{R}^m$.

Where the proof of the above statements are the classical results in matrix analysis (for more details see (G. Allaire and S. M. Kaber, 2008; R. Precup, 2009) and I.A. Rus, 1979).

Definition 1.1:

(T. Gnana Bhaskar, V. Lakshmikantham, 2006). *Let $(X, \leq)$ be a partially ordered set and $F: X \times X \to X$. Mapping F is said to be has the mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for every $x, y \in X$,*

1. for each $x_1, x_2 \in X$, if $x_1 \leq x_2$, then $F(x_1, y) \leq F(x_2, y)$;
2. for each $y_1, y_2 \in X$, if $y_1 \leq y_2$, then $F(x_1, y) \geq F(x_2, y)$.

Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X,d) is a complete metric space. The product space $X \times X$ is endowed with the following partial order:

$$\text{for} \quad (x,y),(u,v) \in X \times X, \quad (u,v) \leq (x,y) \Leftrightarrow x \geq u, \quad y \leq v.$$

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Definition 1.2:

(T. Gnana Bhaskar, V. Lakshmikantham, 2006). *Let* $(X,\leq)$ *be a partially ordered set and* $F: X \times X \to X$. *An element* $(x,y) \in X \times X$ *is said to be a coupled fixed point of the mapping* F *, if* $F(x,y) = x$ *and* $F(y,x) = x$.

Gnana Bhaskar and Lakshmikantham in (T. Gnana Bhaskar, V. Lakshmikantham, 2006), proved the following important Theorem:

Theorem 1.3:

[3, Theorem 2.1]Let $(X,\leq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $F: X \times X \to X$ *be a continuous mapping having the mixed monotone property on* X . *Assume that there exists a* $k \in [0,1)$ *with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u), d(y,v)],$$

for all $x \geq u$ and $y \leq v$. If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(x_0, y_0).$$

Then there exist $x, y \in X$ such that

$$x = F(x,y) \quad \text{and} \quad y = F(y,x).$$

2 Main Results:

Theorem 2.1.:

Let (X,d) *complete generalized metric space, and* $F: X \times X \to X$ *be a continuous mapping having the mixed monotone property on* X . *Assume that there exists* $A \in M_{m\times m}(\mathbb{R}_+),\ A \neq I$ *be a nonzero matrix converging to zero whit:*

$$d(F(x,y), F(u,v)) \leq A[d(x,u) + d(y,v)], \qquad x \geq u, y \leq v.$$

If there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Then, there exists $x, y \in X$ such that $x = F(x,y), y = F(y,x)$.

Proof. Since $x_0 \leq F(x_0, y_0) = x_1$ and $y_0 \geq F(y_0, x_0) = x_1$. Suppose that $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we denote

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2.$$

By this notation, due to the mixed monotone property of F , we have

$$x_2 = F^2(x_0, y_0) = F(x_1, y_1) \geq F(x_0, y_0) = x_1,$$

and

$$y_2 = F^2(y_0, x_0) = F(y_1, x_1) \leq F(y_0, x_0) = y_1.$$

Further, for $n = 1,2,...,$ we let

$$x_{n+1} = F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)),$$

and

$$y_{n+1} == F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)).$$

We can easily verify that

$$x_0 \leq F(x_0, y_0) = x_1 \leq F^2(x_0, y_0) = x_2 \leq ... \leq F^{n+1}(x_0, y_0) \leq ...,$$

and

$$y_0 \geq F(y_0, x_0) = y_1 \geq F^2(y_0, x_0) = y_2 \geq ... \geq F^{n+1}(y_0, x_0) \geq ....$$

Now, we claim that, for $n \in N$,

$$d(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq A^n[d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)], \tag{2.1}$$

and

$$d(F^{n+1}(y_0, x_0), F^n(y_0, x_0)) \leq A^n[d(F(y_0, x_0), y_0) + d(F(x_0, y_0), x_0)]. \tag{2.2}$$

Indeed, for $n=1$, by using of $F(x_0,y_0)\geq x_0$ and $F(y_0,x_0)\leq y_0$, we have

$$d(F^2(x_0,y_0),F(x_0,y_0)) = d(F(F(x_0,y_0),F(y_0,x_0)),F(x_0,y_0))$$
$$\leq A[d(F(x_0,y_0),x_0)+d(F(y_0,x_0))+d(F(y_0,x_0),y_0)].$$

Similarly,

$$d(F^2(y_0,x_0),F(y_0,x_0)) = d(F(F(y_0,x_0),F(x_0,y_0)),F(y_0,x_0))$$
$$= d(F(y_0,x_0),F(F(y_0,x_0),F(x_0,y_0)))$$
$$\leq A[d(F(y_0,x_0),y_0)+d(F(x_0,y_0),x_0)].$$

Then by induction, the relations of (2.1) and (2.2) hold. By $F^{n+1}(x_0,y_0)\geq F^n(x_0,y-0)$ and $F^{n+1}(y_0,x_0)\leq F^n(y_0,x_0)$, we have

$$d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0)) = d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^n(x_0,y_0),F^n(y_0,x_0)))$$
$$\leq A[d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))+d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))]$$
$$\leq A^{n+1}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)].$$

Similarly, we can show that

$$d(F^{n+2}(x_0,y_0),F^{n+1}(y_0,x_0))\leq A^{n+1}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)].$$

This implies that $\{F^n(x_0,y_0)\}$ and $\{F^n(y_0,x_0)\}$ are cauchy sequences in X. Indeed, let $m>n$, then

$$d(F^m(x_0,y_0),F^n(x_0,y_0))\leq d(F^m(x_0,y_0),F^{m-1}(x_0,y_0))$$
$$+\ldots+d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))$$
$$\leq (A^{m-1}+A^{m-2}+\ldots+A^n)[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)]$$
$$= \frac{(A^n-A^m)}{(I-A)}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)]$$
$$< \frac{A^n}{I-A}[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)].$$

Similarly, we can verify $\{F^n(y_0,x_0)\}$ is also cauchy sequence.

Since X is complete metric space, hence there exist $x,y\in X$ such that $\lim_{n\to\infty}F^n(x_0,y_0)=x$ and $\lim_{m\to\infty}F^m(y_0,x_0)=y.$

Finally, we claim that $F(x,y)=x$ and $F(y,x)=y.$ Let $\varepsilon>0$. Since F is continuous at (x,y), for a given $\frac{\varepsilon}{2}>0$, there exists $\delta>0$ such that $d(x,u)+d(y,v)<\delta$ implies that $d(F(x,y),F(u,v))<\frac{\varepsilon}{2}>0.$

Since $F^n(x_0,y_0)\to x$ and $F^n(y_0,x_0)\to y$, for $\eta=min(\frac{\varepsilon}{2},\frac{\delta}{2})$, there exist n_0, m_0 such that for $n\geq n_0$, $m\geq m_0$,

$$d(F^n(x_0,y_0),x)<\eta \quad \text{and} \quad d(F^m(y_0,x_0),y)<\eta.$$

Now, for $n\in N$, where $n\geq \max\{n_0,m_0\}$,

$$d(F(x,y),x) \le d(F(x,y),F^{n+1}(x_0,y_0)) + d(F^{n+1}(x_0,y_0),x)$$
$$= d(F(x,y),F(F^n(x_0,y_0),F^n(y_0,x_0))) + d(F^{n+1}(x_0,y_0),x)$$
$$< \frac{\varepsilon}{2} + \eta \le \varepsilon.$$

This implies that $F(x,y) = x.$ Similarly, we can show that $F(y,x) = y.$

Now we are going to prove that the coupled fixed point is unique, provided that the product space $X \times X$ endowed whit the partial order mentioned earlier has the following property:

(i) Every pair of elements has either a lower bounded or upper bounded.

It is known that this condition is equivalent to:

For every $(x,y),(x^*,y^*) \in X \times X$, there exists $(z_1,z_2) \in X \times X$ that is comparable to (x,y) and (x^*,y^*).

Theorem 2.2.:

Adding condition (i) to the hypothesis of Theorem 2.1, we obtain the uniqueness of coupled fixed point of F.

Proof. If $(x^*,y^*) \in X \times X$ is another coupled fixed point of F, then we show that

$d((x,y),(x^*,y^*)) = 0,$

where $\lim_{n\to\infty} F^n(x_0,y_0) = x$ and $\lim_{m\to\infty} F^m(y_0,x_0) = y.$

We consider two cases:

Case 1: If (x,y) is comparable to (x^*,y^*) whit respect to ordering in $X \times X$, then for every $n \ge 1$, $(F^n(x,y),F^n(y,x)) = (x,y)$ is comparable to $(F^n(x^*,y^*),F^n(y^*,x^*)) = (x^*,y^*).$ Also,

$$d((x,y),(x^*,y^*)) = d(x,x^*) + d(y,y^*)$$
$$= d(F^n(x,y),F^n(x^*,y^*)) + d(F^n(y,x),F^n(y^*,x^*))$$
$$\le A^n d[d((x,y),(x^*,y^*))].$$

But $A \ne I$ this implies that $d((x,y),(x^*,y^*)) = 0$.

Case 2: If $\hat{x} = (x,y)$ is not comparable to $\hat{x}^* = (x^*,y^*),$ then there exists an upper bounded or lower bounded $z = (z_1,z_2) \in X \times X$ of $\hat{x}$, $\hat{x}^*$. Then for all $n = 0,1,2,\dots,$ $F^n(z_1,z_2),F^n(z_2,z_1)$ is comparable to $(F^n(x,y),F^n(y,x)) = (x,y)$ and $(F^n(x^*,y^*),F^n(y^*,x^*)) = (x^*,y^*)$. Therefore we have

$$d((x,y),(x^*,y^*)) \le d((F^n(x,y),F^n(y,x)),(F^n(x^*,y^*),F^n(y^*,x^*))$$
$$\le d(F^n(x,y),F^n(y,x)),(F^n(z_1,z_2),F^n(z_2,z_1))$$
$$+ d((F^n(z_1,z_2),F^n(z_2,z_1)),(F^{(}x^*,y^*),F^n(y^*,x^*)))$$
$$\le A^n[d((x,z_1) + d(y,z_2))] + [d(z_1,x^*) + d(z_2,y^*)].$$

If $n \to \infty$ then $A^n \to 0$, thus $d((x,y),(x^*,y^*)) = 0$.

Theorem 2.3.:

In addition to the hypothesis Theorem 2.1, suppose that every pair of elements of X *has an upper bounded or lower bounded in* X. *Then* $x = y$.

Proof. **Case 1:** If x is comparable to y, then $x = F(x,y)$ is comparable to $y = F(x,y)$ and we have

$d(x,y) = d(F(x,y),F(yh,x)) \le Ad(x,y),$

and since $A \ne I$, this implies that $d(x,y) = 0$.

Case 2: If x is not comparable to y, then there exists an upper bounded or lower bounded of x and y.

Therefore there exists $z \in X$ such that it is comparable to x and y. Suppose that $x \le z$, $y \le z$. Then, we have $F(x,y) \le F(z,y)$, $F(x,y) \ge F(x,z)$, $F(y,x) \le F(z,x)$ and $F(y,x) \ge F(y,z)$.

This yields, by the mixed monotone property of $F^2(x,y) \le F^2(z,y)$, $F^2(y,x) \le F^2(z,x)$, $F^2(x,y) \ge F^2(x,z)$ and $F^2(y,x) \ge F^2(y,z)$.

Similarly relations can be shown to hold for any $n > 2$ too. Now, consider

$$\begin{aligned}
d(x,y) &= d(F^{n+1}(x,y), F^{n+1}(y,x)) \\
&= d(F(F^n(x,y), F^n(y,x))) \\
&= d(F(F^n(x,y), F^n(y,x)), F(F^n(y,x), F^n(x,y))) \\
&\le d(F(F^n(x,y), F^n(y,x)), F(F^n(x,z), F^n(z,x))) \\
&\quad + d(F(F^n(x,z), F^n(z,x)), F(F^n(y,x), F^n(x,y))) \\
&\le d(F(F^n(x,y), F^n(y,x)), F(F^n(x,z), F^n(z,x))) \\
&\quad + d(F(F^n(x,z), F^n(z,x)), F(F^n(z,x), F^n(x,z))) \\
&\quad + d(F(F^n(z,x), F^n(x,z))), F(F^n(y,x), F^n(x,y))).
\end{aligned}$$

By Using of the contractivitiy condition on F we have

$$\begin{aligned}
d(x,y) &\le A[d((F^n(x,y), F^n(x,z)) + d(F^n(y,x), F^n(z,x)) \\
&\quad + d(F^n(x,z), F^n(z,x)) + d(F^n(z,x), F^n(x,z)) \\
&\quad + d((F^n(z,x), F^n(y,x)) + d(F^n(x,z), F^n(x,y))] \\
&\le A[d((F^n(x,y), F^n(x,z)) + d(F^n(x,z), F^n(z,x)) + d(F^n(z,x), F^n(y,x))].
\end{aligned}$$

Thus $d(x,y) \le A^{n+1}[d(x,z) + d(z,y)] \to 0$ as $n \to \infty$, so that $d(x,y) = 0$.

Alternatively, if we know that elements x_0, y_0 are such that $x_0 \le y_0$, then we can also demonstrate that the components x and y of the coupled fixed point are indeed the same.

Theorem 2.4.:

In addition to the hypothesis Theorem 2.1, suppose that x_0, y_0 *in* X *are comparable. Then* $x = y$.

Proof. Let x_0 and y_0 be as Theorem 2.1. We claim that, for all $n \in N$, $x_n \le y_n$. We show that this by induction. Indeed, by the mixed monotone property of F we have

$$x_1 = F(x_0, y_0) \le F(y_0, x_0) = y_1.$$

Assume that $x_n \le y_n$ for some n. Now, consider

$$x_{n+1} = F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) = F(x_n, y_n) \le F(y_n, x_n) = y_{n+1}.$$

Hence, $x_n \le y_n$ for all n. For a given $\varepsilon > 0$, there exists $n_0 \in N$ such that $d(x, F^n(x_0, y_0)) < \dfrac{\varepsilon}{4}$ and $d(x, F^n(y_0, x_0)) < \dfrac{\varepsilon}{4}$ for all $n \ge n_0$, and

$$
\begin{aligned}
d(x,y) &\le d(x, F^{n+1}(x_0,y_0)) + d(F^{n+1}(x_0,y_0), y) \\
&\le d(x, F^{n+1}(x_0,y_0)) + d(F^{n+1}(x_0,y_0), F^{n+1}(y_0,x_0)) + d(F^{n+1}(y_0,x_0), y) \\
&< \frac{\varepsilon}{4} + d(F(F^n(x_0,y_0), F^n(y_0,x_0)), F(F^n(y_0,x_0), F^n(x_0,y_0))) + \frac{\varepsilon}{4} \\
&\le \frac{\varepsilon}{2} + Ad((F^n(y_0,x_0), F^n(x_0,y_0)) \\
&\le \frac{\varepsilon}{2} + A[d((F^n(y_0,x_0), y) + d(x,y) + d(x, F^n(x_0,y_0))] \\
&\le \varepsilon + Ad(x,y).
\end{aligned}
$$

This implies that $(I-A)d(x,y) < \varepsilon$, which in turn leads to $d(x,y)=0$, hence we have $x=y$. Similarly, if $x_0 \ge y_0$, then it is possible to show that $x_n \ge y_n$ for all n and therefore $d(x,y)=0$.

REFERENCES

Allaire, G. and S.M. Kaber, 2008. *Numerical Linear Algebra*, vol. 55 of Texts in Applied Mathematics, Springer-New York.

Du, W.-S., 2010. *Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi–Takahashi's condition in quasiordered metric spaces*, Fixed Point Theory Appl. 2010 (2010) Article ID 876372, 9.

Gnana Bhaskar, T., V. Lakshmikantham, 2006. *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., 65: 1379-1393.

Filip, A.D. and Petru s el, 2010. *Fixed point theorems on spaces endowed with vector-valued metrics*, Fixed Point Theory and Applications, vol. Article ID 281381, 15 pages, 2010.

Precup, R., 2009.*The role of matrices that are convergent to zero in the study of semilinear operator systems*, Mathematical and Computer Modelling, 49(3-4): 703-708.

Rus, I.A., 1979. *Principles and Applications of the Fixed Point Theory*, Dacia, Cluj-Napoca, Romania,

Varga, R.S., 2000. *Matrix Iterative Analysis*, vol. 27 of Springer Series in Computational Mathematics, Springer-Berlin,