Australian Journal of Basic and Applied Sciences, 6(2): 175-181, 2012
ISSN 1991-8178

Solution of The Higher-Order Integro-Differential Equations by Variational Iteration Method

[1]Neda Najafzadeh, [2]Mehrasa Ayatollahi, [3]Sohrab Effati

[1,2]Department of Mathematics, Payame Noor University (PNU), PO BOX 19395-6697 Tehran, Iran.
[3]Department of Mathematics, Ferdowsi University, Mashad, Iran.

Abstract: This paper applies the variational iteration method for solving higher order integro differential equations(IDES). We have shown that higher order integro differential equations can be transformed into a system of integral equations,wich can be solved by using variational iteration method. The analytical results of the equations havebeen obtained in terms of convergent series with easily computable components. Several examples are given to illustrate the efficiency and implementation of the method.

Key words: integro differential equations, variational iteration, integral equations.

INTRODUCTION

In recent years, there has been agrowing interest in the Integro-Differential Equations (IDEs). IDEs play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics. The higher-order integro differential equations arise in mathematical, applied and engineering sciences, astrophysics, solid state physics, astronomy, fluid dynamics heam theory, fiber optics and chemical reaction-diffusion models; (R.P. Agarwal, 1986; R.P. Agarwal, 1983; L. Hashim, 2006; J.H. He, 2000; J. Morchalo, 1975; A.M. Wazwaz, 2000) and the references therein. The mentioned integro-differential equations are usually difficult to solve analytically; so a numericalmethod is required.

The variational iteration method (VIM) is a simple and yet powerful method for solving a wide class of nonlinear problems, first envisioned by He (1998; 2000; 2006; 2003). The VIM has successfully been applied to many situations. For example He (1998) solved the classical Blasius' equation using VIM. He (1999) used VIM to give approximate solutions for some well-known nonlinear problems. He (2000) used VIM to solveautonomous ordinary differential systems. He (2003) coupled the iteration method with the perturbation method to solve the well-known Blasius equation. He (2006) solved strongly nonlinear equations using VIM. Soliman (2005) applied the VIM to solve the KDV-Burger's and lax's seventh-order KDV equations. The VIM has recently been applied for solving nonlinear coagulation problem with mass loss by Abulwafa *et al*. (2005). The VIM has been applied for solving nonlinear differential equations of fractional order by Odibat *et al*. (2006). Bildik *et al*. (2006) used VIM for solving differnt types of nonlinear partial differential equations. Dehghan and Tatari (2006) employed VIM to solve a Fokker-Planck equation. Wazwaz (2006) presented a comparative study between the variational iteration method and Adomian decomposition method. Tamer *et al*. (2006) introduced a modification of VIM. Abbasandy (2006) solved one example of the quadratic Riccati differential equation by He's VIM by using Adomian's polynomials.

In the paper, we apply the variational iteration method for solving the higher-orderintegro differential equations. It is worth mentioning that this method was first considered by Inokuti *et al*. (1978). The basic motivation of this paper is to apply the variational iteration method to solve a system of integral equations. It is shown that the method provides the solution in a rapid convergent series. The variational iteration method has been shown (Aslam Noor, M., 2008; Aslam Noor, M., 2008; Inokuti, M., 1978; Aslam Noor, M., 2007; Aslam Noor, M., 2007; J.H. He, 1999; J.H. He, 2006) to solve effectively, easily and accurately a larg class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. It can be shown that higher-order integro differential equations are equivalent to the system of integral equations can be solved efficiently using the variationaliteration method.

Variational Iteration Technique:

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu+Nu= g (X) \tag{1}$$

Corresponding Author: Neda Najafzadeh, Department of Mathematics,Payame NoorUniversity(PNU), PO BOX 19395-6697 Tehran, Iran.
E-mail: mn_najafzadeh@yahoo.com

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method (Aslam Noor, M., 2008; Aslam Noor, M., 2008), we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\, (Lu_n(s) + N\tilde{u}_n(s) - g(s))ds \tag{2}$$

Where A is Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, is considered as a restricted variation. That is δ= 0; (2)is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in (Aslam Noor, M., 2008; Aslam Noor, M., 2008; (Aslam Noor, M., 2007; Aslam Noor, M., 2007). For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$\acute{x}_i(t) = f_i(t, x_i),\ \tilde{U}_n \text{ i= 1,2,3,..., n,} \tag{3}$$

Subject to the boundary conditions:x_i (0)= ci, $\tilde{U}_n$ i=1,2,3,…,n

To solve the system by means of the variation iteration method, we rewrite thesystem(3) in the following form:

$$\acute{x}_i(t) = f_i(x_i) + g_i(t) \text{i=1,2,3,…,n} \tag{4}$$

Subject to the boundary conditions:x_i (0)=c_i,i=1,2,3,…,n and g_i is defined in (1).

The correct functional for the nonlinear system (3) can be approximated as:

$$x_1^{(k+1)}(t) = x_1^{(0)}(t) + \int_0^t \lambda_1 \left(\acute{x}_1^{(k)}(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_1(T)\right) dT,$$

$$x_2^{(k+1)}(t) = x_2^{(0)}(t) + \int_0^t \lambda_2 \left(\acute{x}_2^{(k)}(T), f_2\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_2(T)\right) dT,$$

.
.
.

$$x_n^{(k+1)}(t) = x_n^{(0)}(t) + \int_0^t \lambda_n \left(\acute{x}_n^{(k)}(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_n(T)\right) dT,$$

Whereλ_i=$\pm$ 1 , i= 1,2,3, … , n are Lagrange multipliers $\tilde{X}_1, \dots, \tilde{X}_n$ denote the restricted variations.

Forλ_i= 1 ,i= 1,2,3,…,n,we have the following iterative schemes:

$$x_1^{(k+1)}(t) = x_1^{(0)}(t) + \int_0^t \left(\acute{x}_1^{(k)}(T), f_1\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_1(T)\right) dT,$$

$$x_2^{(k+1)}(t) = x_2^{(0)}(t) + \int_0^t \left(\acute{x}_2^{(k)}(T), f_2\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_2(T)\right) dT,$$

.
.
.

$$x_n^{(k+1)}(t) = x_n^{(0)}(t) + \int_0^t \left(\acute{x}_n^{(k)}(T), f_n\left(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)\right) - g_n(T)\right) dT,$$

If we start with the initial approximationx_i (0)=c_i,i=1,2,3,…,n,then theapproximations can be completely determined. Finally we approximate the solution:

x_i (t)=$\lim_{n\to\infty} x_i^{(n)}(t)$ by the nth term$x_i^{(n)}(t)$i= 1,2,3,…,n

Solution Of Integro Differential Equations:

We consider the linear boundary value problem for the higher-order integro differential equation

$$y^{(n)}(x) = g(x) + f(x)y(x) + \lambda \int_0^x K(x,t)\, y(t)\, dt, \tag{5}$$

With initial conditions

$$y(0) = \alpha_0,\ \dot{y}(0) = \alpha_1, \ldots, y^{(n-1)}(0) = \alpha_{n-1}, \tag{6}$$

We consider the transformation

$$y(x) = y_1(x), \frac{dy}{dx} = y_2(x), \frac{d^2y}{dx^2} = y_3(x) \ldots, \frac{d^{(n-1)}y}{dx^{(n-1)}} = y\ (x) \tag{7}$$

We rewrite the above higher-order boundary value problem as a system of differentialequations:

$$\frac{dy_1}{dx} = y_2(x),$$

$$\frac{dy_2}{dx} = y_3(x),$$

.
.
.

$$\frac{dy_n}{dx} = g(x) + f(x)y_1(x) + \lambda \int_0^x K(x,t)y_1(t)dt,$$

With initial conditions:

$y_1^{(0)}(x) = \alpha_o,\ y_2^{(0)}(x) = \alpha_1, \ldots, y_n^{(0)}(x) = \alpha_{n-1}$

The above system of differential equations can be written as a system of integralequations with Lagrange multipliers $\lambda_i = 1$,i= 1,2,3,…,n

$$y_1^{(k+1)}(x) = y_1^{(o)}(x) + \int_o^x y_2^{(k)}(s)ds,$$

$$y_2^{(k+1)}(x) = y_2^{(o)}(x) + \int_o^x y_3^{(k)}(s)ds,$$

.
.
.

$$y_n^{(k+1)}(x) = y_1^{(o)}(x) + \int_o^x \left[g(s) + f(s)y_1^{(k)}(s) + \lambda \int_o^s k(s,t)y_1^{(k)}(t)dt\right] ds$$

For example with k = 0 we obtain

$$y_1^{(1)}(x) = \alpha_o + \int_o^x \alpha_1 ds = \alpha_o + \alpha_1 x,$$

$$y_2^{(1)}(x) = \alpha_1 + \int_o^x \alpha_2 ds = \alpha_1 + \alpha_2 x,$$

.
.
.

$$y_n^{(1)}(x) = \alpha_{n-1} + \int_o^x \left[g(s) + f(s)\alpha_o + \lambda \int_o^x k(s,t)\alpha_o dt\right] ds.$$

Applications:
Example1:

Consider the following integro differential equation

$$y^{iv}(x) = x(1+\exp(x)) + 3\exp(x) + y(x) - \int_o^x y(t)dt$$

With boundary conditions:

$y(0)=1, y'(0)=1, y(1)=1+e, y''(1)=2e$

The exact solution for this problem is: $y(x) = 1 + xe^x$
Using the transformation $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_4 = y'''$

We rewrite the above problem as a system of differential equations:

$$\begin{cases} \frac{dy_1}{dx} = y_2(x), \\ \frac{dy_2}{dx} = y_3(x), \\ \frac{dy_3}{dx} = y_4(x), \\ \frac{dy_4}{dx} = x(1+\exp(x)) + 3\exp(x) + y_1(x) - \int_o^x y_1(t)dt. \end{cases}$$

The above system of differential equations can be written as asystem of integralequations with Lagrange multipliers$\lambda_i = 1$ $i = 1,2,\ldots,n$

$$y_1^{(k+1)}(x) = y_1^{(o)}(x) + \int_o^x y_2^{(k)}(s)ds,$$

$$y_2^{(k+1)}(x) = y_2^{(o)}(x) + \int_o^x y_3^{(k)}(s)ds,$$

$$y_3^{(k+1)}(x) = y_3^{(o)}(x) + \int_o^x y_4^{(k)}(s)ds,$$

$$y_4^{(k+1)}(x) = y_4^{(o)}(x) + \int_o^x \left[s(1+\exp(s)) + 3\exp(s) + y_1^{(k)}(s) - \int_o^s y_1^{(k)}(t)dt\right]ds.$$

With$y_1^{(0)}=1, y_2^{(0)}=1, y_3^{(0)}=A, y_4^{(0)}=B$

Consequently, we obtain the following approximations:

$y_1^{(1)}=1+x$ $\quad$ $y_2^{(1)}=1+Ax$

$y_3^{(1)}=A+Bx$ $\quad$ $y_4^{(1)}=B+x\exp(x)+2\exp(x)+x+1$

$$y_1^{(2)} = 1 + x + \frac{Ax^2}{2}$$

$$y_2^{(2)} = 1 + Ax + \frac{Bx^2}{2}$$

$$y_3^{(2)} = A + Bx + xexp(x) + 3\exp(x) + \frac{x^2}{2} + x$$

$$y_4^{(2)} = B + xexp(x) + 4\exp(x) + x + \frac{x^2}{2} - \frac{x^3}{6}$$

.
.
.

Using the boundary conditions at x = 1, we have
A = 1.999999953, B = 3.00000151

Table 4.1.1: Numerical comparison of the exact solution versus the approximation method.

Error	VIM	Exact solution	x
0.0000	1.00000000	1.00000000	0
2.0E - 9	1.11105170	1.11105170	0.1
1.5E - 9	1.24428054	1.24428055	0.2
4.0E - 8	1.40495760	1.40495764	0.3
2.1E - 8	1.59672985	1.59672987	0.4
3.2E - 8	1.82436060	1.82436063	0.5
1.2E - 6	2.09327006	2.09327128	0.6
1.4E - 6	2.40962585	2.40962689	0.7
2.0E - 6	2.78043070	2.78043274	0.8
1.9E - 7	3.21364261	3.21364280	0.9
2.0E - 9	3.71828180	3.71828182	1

4.2 Example2:

Consider the following integro differential equation

$$y^{iv}(x) = 1 + \int_{o}^{x} \exp(-x)\, y^2(t)dt$$

With boundary conditions:

$$y(o)=1 , y'(o)=1 , y(1)=e , y''(1)=e$$

The exact solution for this problem is: y(x) = exp(x).
Using the transformation $y_1 = y$, $y_2 = y'$, $y_3 = y''$, $y_4 = y'''$
We rewrite the above problem as a system of differential equations:

$$\begin{cases} \dfrac{dy_1}{dx} = y_2(x), \\ \dfrac{dy_2}{dx} = y_3(x), \\ \dfrac{dy_3}{dx} = y_4(x), \\ \dfrac{dy_4}{dx} = 1 + \int_{o}^{x} \exp(-x)\, y_1^2(t)dt. \end{cases}$$

The above system of differential equations can be written as asystem of integralequations with Lagrange multipliers $\lambda_i = 1$ i= 1,2,…,n

$$y_1^{(k+1)}(x) = y_1^{(o)}(x) + \int_{o}^{r} y_2^{(k)}(s)ds,$$

$$y_2^{(k+1)}(x) = y_2^{(o)}(x) + \int_{o}^{x} y_3^{(k)}(s)ds,$$

$$y_3^{(k+1)}(x) = y_3^{(o)}(x) + \int_{o}^{x} y_4^{(k)}(s)ds,$$

$$y_4^{(k+1)}(x) = y_4^{(o)}(x) + \int_{o}^{x} \left[1 + \int_{o}^{s} \exp(-s)\, y_1^{(k)}(t))^2 dt\right] ds,$$

With$y_1^{(0)}=1$, $y_2^{(0)}=1$,$y_3^{(0)}=A$, $y_4^{(0)}=B$
Consequently, we obtain the following approximations:

$y_1^{(1)}=1+x$ $\quad$ $y_2^{(1)}=1+Ax$

$y_3^{(1)}=A+Bx$ $\quad$ $y_4^{(1)} =B+\exp(-x)+2x-1$

$$y_1^{(2)} = 1 + x + \frac{Ax^2}{2}$$

$$y_2^{(2)} = 1 + Ax + \frac{Bx^2}{2}$$

$$y_3^{(2)} = A + Bx - exp(-x) + x^2 + 1$$

$$y_4^{(2)} = B + x^2 \exp(-x) + 6xexp(-x) + 11\exp(-x) + 6x - 6$$

.
.
.

Using the boundary conditions at x = 1, we have

Table 4.2.1: Numerical comparison of the exact solution versus the approximation method. A = 0.99708595 , B = 1.0109940.

Error	VIM	Exact solution	x
0.0000	1.00000000	1.00000000	0
1.1E - 8	1.10515817	1.10515818	0.1
2.0E - 8	1.22140225	1.22140227	0.2
1.2E - 6	1.34985778	1.34985880	0.3
3.2E -7	1.49182437	1.49182469	0.4
9.3E -7	1.64872034	1.64872127	0.5
1.0E - 6	1.82211777	1.82211880	0.6
1.9E - 6	2.01375078	2.01375270	0.7
9.0E - 8	2.22554083	2.22554092	0.8
1.0E - 8	2.45960310	2.45960311	0.9
2.9E -7	2.71828153	2.71828182	1

Concolusion:

In this paper, variational iteration method was successfully deduced for solving high-orderintegrodiffrential equations. The numerical results in the tables show that the present method provides highly accurate numerical solutions for solving this type of equations.

REFERENCES

Abbasbandy, S., 2006. A new application of He's variation iteration method for quadratic Riccati differential equation by using Adomin's polynomials, J. Comput. Appl. Math., 07. 012.

Abulwafa, E.M., M.A. Abdou and A.A. Mahmoud , 2005. The solution of nonlinear coagulation problem with mass loss. Chaos Solitons Fractals, 08. 044.

Agarwal, R.P., 1983. Boundary value problems for higher order integro-differential equationsNonlinear Analysis, Theory Math. Appl., 7(3): 259-270.

Agarwal, R.P., 1986. Boundary value problems for higher order differential equations, World scientific, Singapore.

Aslam Noor M, Mohydu-Din ST., 2008. Modified variational iteration method for heat and wave-like equations. Act Appl Math.

Aslam Noor M,Mohydu-Din ST., 2008. Variationaliteration method for solving initial and boundary value problems of Bratu-type,applications.Appl Math., 3: 89-99.

Aslam Noor, M., S.T. Mohyud-Din, 2007. An efficient method for fourth-order boundary value problems. Comput Math Appl., 54: 1101-11.

Aslam Noor, M., S.T. Mohyud-Din, 2007. Variational iteration technique for solving higher order boundary value problems.Appl Math Comput., 189: 1929-42.

Bildik, N. and A. Konuralp, 2006. The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, Int.J. Nonlinear.sci.Numer.Simul., 7: 65-70.

Dehghan, M. and M. Tatari, 2006. The use of He's variational iteration method for solving a Fokker-Planck equation. Phys. Scr., 74: 310-316.

El-Zoheiry, H., T.A. Abassy and M.A. El-Tawil, 2006. Toward a modified variational iteration method. J. Comput. Appl. Math., 07.019.

Hashim, L., 2006. Adomian's decomposition method for solving boundary value problems for fourth-order integro-differential equations,J.Comput. Appl. Math., 193: 658-664.

He, J.H., 1998. Approximate analytical solution of Blasius equation. Commun.Nonlin. Sci. Numer. Simul., 3: 260-263.

He, J.H., 1999. Variational iteration method-A kind of non-linear analytical technique:someexamples,Int.J.Nonlinear Mech., 34(4): 699-708.

He, J.H., 2000. Variationaliteration method for autonomous ordinary differential systems. Appl. Math. Comput., 114: 115-123.

He, J.H., 2003. A simple perturbation approach to Blasius equation. Appl. Math. Comput., 140: 217-222.

He, J.H., 2006. Some asymptotic methods for strongly nonlinear equations. Int. J. Modern Phys., B20: 1141-1199.

He, J.H., 2006. Variational iteration method-Some recent results and new interpretations,J. Comput.Appl. Math.doi:10. 10 16jj.cam. 2006. 07. OOg.

Inokuti, M., H. Sekine, T. Mura, 1978. General using of lagrange multiplier in nonlinear mathematical physics.ln:variational methods in the mechanics of solids. New York:Pergamon Press, pp: 56-62.

J.H. He, 2000. A coupling method of homotopytecnique and perturbation technique for nonlinear problems,Int. J.NonlinearMech., 35(1): 115-123.

Morchalo, J., 1975. On two point boundary value problem for integro differential equation of higher order, Fasc.Math., 9: 77-96.

Odibat, Z.M. and S. Momani, 2006. Application of variationaliteratin method to nonlinear differential equations of fractional order. Int. J. Nonlinear Sci. Numer.Simul., 7: 27-34.

Soliman, A.A., 2005. A numerical simulation and explicit solutions of KdV-Burger's and Lax's seventh-order KdV equations. Chaos solitons Fractals, 08.054.

Wazwaz, A.M., 2000. A reliable algorithm for solving boundary value problems for higher order integro differential equations, Appl. Math.comput. 118: 327-342.

Wazwaz, A.M., 2006. A comparison between the variationaliteration method and adomin decomposition method. J. Comput. Appl. Math., 07.018.