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On The Approximation Of The Spectral Expansions Of Distributions In The Weak Topology.

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Abstract: In this paper we study approximations of distributions on a smooth manifold. We obtain sufficient conditions of summation of the series in the weak topology. As applications we consider spectral expansions connected with elliptic partial differential operators.

Key words: Approximations of distributions, spectral expansions, functional spaces, weak topology.

Preliminaries:

Let Ω N - dimensional manifold, i.e. a Housdorff topological space in which each point has a neighborhood, which is homeomorphic to an open N - dimensional sphere. We suppose that Ω is a smooth manifold, which means: for each point $x \in \Omega$ there exists a neighborhood $O(x) \subset \Omega$ and infinite time differentiable mapping f_x of this neighborhood on a certain open set $O(x)$ of the space R^N and these mappings are compatible in the intersection of neighborhoods, that means if $O(x_1) \cap O(x_2) \neq \varnothing$ then a mapping

$$f_{x_1} f_{x_2}^{-1} : f_{x_2} O(x_1) \cap O(x_2) \rightarrow f_{x_1} O(x_1) \cap O(x_2)$$

is infinite time differentiable for all x_1 and x_2. We also suppose that Ω is a paracompact manifold, i.e. a topological space in which every open cover admits a locally finite open refinement (each point in the manifold has a neighborhood that intersects only finitely number of the member sets in the cover).

By $C^\infty(\Omega)$ denote a space of all smooth (infinite differentiable) functions on manifold Ω: a function φ defined on manifold Ω belongs to $C^\infty(\Omega)$ if each function $\varphi(f_x^{-1}(y))$ belongs to the space $C^\infty(\mathcal{O}(x))$. Denote by $C_0^\infty(\Omega)$ a set of those functions from $C^\infty(\Omega)$ that have a compact support in Ω. For simplicity, we consider that all functions below are real.

In $C_0^\infty(\Omega)$ and $C^\infty(\Omega)$ we define topologies. By α denote a multiindex, i.e. N - dimensional vector with non negative integer components $\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_N)$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$ called length of multiindex. Then for any other vector $\xi \in R^N$ we define its α power as $\xi^\alpha = \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdots\cdots \xi_N^{\alpha_N}$, where ξ_j - components of ξ. Let $\nabla_j = \dfrac{\partial}{\partial x_j}$ are components of the gradient vector (differential vector operator) $\nabla = (\nabla_1, \nabla_2, \ldots\ldots, \nabla_N)$. Thus replacing ξ by ∇ and using a definition of its α power obtain $\nabla^\alpha = \nabla_1^{\alpha_1} \cdot \nabla_2^{\alpha_2} \cdots\cdots \nabla_N^{\alpha_N}$.

Definition 1. *Let $\varphi_n(x)$ a sequence of functions from $C_0^\infty(\Omega)$. We will say that a sequence φ_n convergence to $\varphi \in C_0^\infty(\Omega)$ in topology of the space $C_0^\infty(\Omega)$ if*

- *there exists a compact set K from Ω such that $\varphi_n(x) = 0$ for all $x \in \Omega \setminus K$ and all numbers n,*

- *for any multiindex α a sequence $\nabla^\alpha \varphi_n(f_x^{-1}(y))$ converges to $\nabla^\alpha \varphi(f_x^{-1}(y))$ uniformly on each compact subset of $\mathcal{O}(x)$ when $n \to \infty$.*

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The space $C_0^\infty(\Omega)$, equipped with this topology denote by $\mathcal{D}(\Omega)$.

Definition 2:

Linear continuous functional f on $\mathcal{D}(\Omega)$ is called a distribution.

A set of all distributions denote by $\mathcal{D}'(\Omega)$. Value of a distribution $f \in \mathcal{D}'(\Omega)$ on a function φ from $\mathcal{D}(\Omega)$ define as $f(\varphi) = < f, \varphi >$. Thus from definition 2 it follows that if $f \in \mathcal{D}'(\Omega)$ and a sequence $\varphi_n \in \mathcal{D}(\Omega)$ converges to zero in topology of the space $\mathcal{D}(\Omega)$, then a sequence of numbers $f(\varphi_n) = < f, \varphi_n >$ will also converge to zero.

The space $\mathcal{D}'(\Omega)$ we consider as a linear topological space with the weak topology, i.e. we say that a sequence of distributions f_n converges to f in topology $\mathcal{D}'(\Omega)$, if $f_n(\varphi) \to f(\varphi)$ for any function φ from $\mathcal{D}(\Omega)$.

Let $\Omega^1 \subseteq \Omega$ and $f \in \mathcal{D}'(\Omega)$. We say that the distribution f is equal zero in subdomain Ω^1, if for arbitrary φ from $\mathcal{D}(\Omega)$ with support in Ω^1, it is true that $< f, \varphi > = 0$. The least closed set outside of which a distribution f is equal zero, is called a support of the distribution f and denotes as $suppf$.

In the construction of the space of distributions $\mathcal{D}'(\Omega)$ as an initial space we used the space of all smooth functions with compact support $C_0^\infty(\Omega)$. If instead of $C_0^\infty(\Omega)$ as an initial space we consider the space $C^\infty(\Omega)$ then we obtain a new space of distributions. For that first introduce a topology in the space $C^\infty(\Omega)$. Topology in this space can be defined by system of semi norms

$$P_{K,\alpha}(\varphi) = \max_{x \in K} \sup_{y \in \mathcal{O}(x), \mathcal{O}(x)} \left| D^\alpha \varphi\left(f_x^{-1}(y)\right)\right|$$

where K is a compact subset of Ω and α is a multiindex.

The space $C^\infty(\Omega)$ with this topology denote by $\mathcal{E}(\Omega)$. In the space $\mathcal{E}(\Omega)$ we define convergence as following

Definition 3:

We say that a sequence $\varphi_n \in C^\infty(\Omega)$ converges to a function $\varphi \in C^\infty(\Omega)$ in topology $\mathcal{E}(\Omega)$, if for any multiindex α and any compact set $K \subset \Omega$ we have $P_{K,\alpha}(\varphi - \varphi_n) \to 0$, when $n \to \infty$.

Corresponding space of distributions (linear continuous functional on $\mathcal{E}(\Omega)$) denote by $\mathcal{E}'(\Omega)$. Obviously we have $\mathcal{E}'(\Omega) \subset \mathcal{D}'(\Omega)$. The space $\mathcal{E}'(\Omega)$ consists of distributions from $\mathcal{D}'(\Omega)$ with compact support in Ω (M. Reed, B. Simon).

Approximation In The Weak Topology.:

Denote by $H = L_2(\Omega)$ a Hilbert space of square integrable functions. Let $\varphi_n(x), n = 1,2,3\ldots$, a complete orthonormal system of smooth (infinite time differentiable) functions in the space H. Thus, we can consider an action of a distribution of f from $\mathcal{E}'(\Omega)$ on a basis function $\varphi_n(x)$ and denote it as $f_n =< f, \varphi_n >$. Then we call f_n n-th coefficient of the distribution f from $\mathcal{E}'(\Omega)$ by orthonormal basis φ_n in H. Thus in this way we define decomposition of a distribution with compact support by given orthonormal system as following relation

$$f : \sum_{n=1}^{\infty} f_n \, \varphi_n(x) \tag{1}$$

We study relation (1) in the weak topology. In the right side of (1) all terms of the series belong to $\mathcal{E}(\Omega)$ and in the left side we have a distribution from the space $\mathcal{E}'(\Omega)$. A distribution f in (1) can be a singular.

For example, the Dirac delta function which also belongs to the space $E'(\Omega)$. Singularity of the distribution f effects on its coefficients f_n and convergence of the series (1) in topologies of the spaces $D(\Omega)$ or $E(\Omega)$, even in subsets of Ω where it coincides with a smooth function, hardly to be expected without some regularization (Abdumalik Rakhimov, Anvarjon Ahmedov and Hishamuddin Zainuddin 2011, Sh.A Alimov and A.A. Rakhimov 1996, A.A. Rakhimov 2000). In case of Dirac's delta function series (1) may diverge even at the points that is not in its support. However, relation (1) still can be considered in sense of topology of distributions.

Note that for any $f \in H$ series (1) always converges in the topology of the Hilbert space H. But this is not true for any f from $E'(\Omega)$. Nevertheless, we can consider approximation in topology of $D'(\Omega)$ if $f \in E'(\Omega)$.

Theorem 1:

If for any ψ from the space $D(\Omega)$ series (1) converges to ψ in topology of the space $\mathcal{E}(\Omega)$, then series (1) converges to f in topology of the space $D'(\Omega)$ for any f from $E'(\Omega)$.

Proof. Let's $f \in E'(\Omega)$. Then partial sums of the series in the right side of (1) we can write as action of the distribution f to a function $\Theta(x, y, n) = \sum_{k=1}^{n} \varphi_k(x)\varphi_k(y)$ which is an element of the space $E(\Omega \times \Omega)$ and the distribution f acts on $\Theta(x, y, n)$ by second argument. Thus partial sums of (1) we write as following

$$E_n f(x) = < f, \Theta > . \tag{2}$$

Topology in the space $E(\Omega \times \Omega)$ can be constructed similarly as in $E(\Omega)$. Note that the function $E_n f(x)$ in (2) also belongs to $E(\Omega)$ for any $f \in E'(\Omega)$. Thus, we can consider $E_n f(x)$ as an element of $D'(\Omega)$ because $E(\Omega) \subset D'(\Omega)$.

Let's consider a sequence of distributions $g(n) = f - E_n f(x)$. We should prove that $g(n) \to 0$ as $n \to \infty$ in topology of the space $D'(\Omega)$. Let ψ is an arbitrary function from $D(\Omega)$. Then we have

$$< g(n), \psi > = < f - E_n f(x), \psi > = < f - < f, \Theta >, \psi > =$$

$$= < f, \psi > - < f, < \psi, \Theta >> = < f, \psi > - < f, E_n\psi > =$$

$$= < f, \psi - E_n\psi > .$$

Note that a sequence of functions $\omega_n(x) = \psi(x) - E_n\psi(x)$ belongs to the space $E(\Omega)$ and according conditions of the theorem above for any multiindex α and any compact set $K \subset \Omega$ we have

$$P_{K,\alpha}(\omega_n) \to 0.$$

As we mentioned above any element of the space $E'(\Omega)$ has a compact support in Ω. Thus if manifold Ω is compact itself (topologically compact), then $E(\Omega) = D(\Omega)$ and therefore $E'(\Omega) = D'(\Omega)$. Then (4) means convergence of the sequence $\omega_n(x)$ in topology of the space $D(\Omega)$. Thus, from (3) by continuity of the distribution f from $D'(\Omega)$ we obtain that the sequence of distributions $g(n)$ converges to zero in topology of the space $D'(\Omega)$. This proves the theorem in this case.

If manifold Ω is not compact, then there is a compact set $K_0 \subset \Omega$ such that $supp\ f \subset K_0 \subset \Omega$. Moreover, this compact set can be chosen such that there are Ω_1 and Ω_2 noncompact submanifolds of the manifold Ω and compact K_1 such that $supp f \subset \Omega_1 \subset K_1 \subset \Omega_2 \subset K_0$. Then for any function $\psi(x)$ from the space $E(\Omega)$ with support in $\Omega \setminus \Omega_1$ we have $f(\psi) = < f, \psi > = 0$. Therefore if $\psi(x)$ from

the space $\mathcal{E}(\Omega)$ coincides with function ψ_1 from $\mathrm{D}(\Omega)$ in the domain Ω_1, then $< f, \psi > = < f, \psi_1 >$. Thus, we denote by $\omega_n^*(x)$ a function from $\mathrm{D}(\Omega)$ that coincides with function $\omega_n(x)$ in Ω_1 and has a support in Ω_2 and we have $< f, \omega_n(x) > = < f, \omega_n^*(x) >$. Thus
$< g(n), \psi > = < f, \omega_n > = < f, \omega_n^* >$.

Then taking into account that from (4) it also follows that $\omega_n^* \to 0$ in topology of the space $\mathrm{D}(\Omega)$ and due to continuity of the distribution f
$< f, \omega_n^* > \to 0.$

Hence, obtain $< g(n), \psi > \to 0$ when $n \to \infty$ for any $\psi(x) \in \mathrm{E}(\Omega)$. This means convergence of series (1) to $f \in \mathrm{E}'(\Omega)$ in topology of the space $\mathrm{D}'(\Omega)$. Theorem 1 proved.

Below we consider two particular applications of theorem 1.

Spectral expansions of elliptic pdo:

Let Ω a domain in R^N with smooth boundary $\partial\Omega$. Let the functions $a_\alpha(x)$ belong to $\mathrm{E}(\Omega)$, where α is a multiindex. Denote by $A(x, \nabla) = \sum_{|\alpha| \le 2m} a_\alpha(x) \nabla^\alpha$ positive elliptic and symmetric differential operator with domain of definition $\mathrm{D}(\Omega)$. It is well known (Sh. A. Alimov, V. A. Il'in, E. M. Nikishin, 1976), that this operator has self-adjoin extension $\hat{A}$ in Hilbert space H and let $\{E_\lambda\}$ a spectral family of projectors in H answering to the operator $\hat{A}$. Projectors E_λ are integral operators in H with kernels $\Theta(x, y, \lambda)$ that belong to the space $\mathrm{E}(\Omega \times \Omega)$. The kernel Θ is called a spectral function of the operator $\hat{A}$.

Define a spectral expansion of a distribution $f \in \mathrm{E}'(\Omega)$ by its action to the spectral function $\Theta(x, y, \lambda)$ on its second argument and denote it as $E_\lambda f(x) = < f, \Theta(x, y, \lambda) >$.

If we prove that the spectral expansions of functions from $\mathrm{D}(\Omega)$ converge in topology of the space $\mathrm{E}(\Omega)$ then from theorem 1 it follows that the spectral expansions $E_\lambda f$ converges to f in topology of the space $\mathrm{D}'(\Omega)$ for any distribution $f \in \mathrm{E}'(\Omega)$.

For any integer number j by $H^j(\Omega)$ denote Sobolev's spaces on manifold Ω (H. Triebel). Let $\varphi \in \mathrm{D}(\Omega)$. Denote $\psi(\lambda) = (E_\lambda - I)\varphi$. Note that operators E_λ and $\hat{A}^k$ are commutate for any number $k = 0,1,2,\ldots$ and we have $\hat{A}^k \varphi \in \mathcal{D}(\Omega) \subset H$. Therefore

$$\hat{A}^k \psi(\lambda) = \hat{A}^k (E_\lambda - I)\varphi = (E_\lambda - I)\hat{A}^k \varphi \to 0$$

in topology of the space H when $\lambda \to \infty$. Then $\psi(\lambda) \to 0$ in topology of the Sobolev spaces H^{km} for any $k = 0,1,2,\ldots$. Which means $E_\lambda \varphi \to \varphi$ in topology of H^{km} for $\varphi \in \mathrm{D}(\Omega)$ and any k. For any number n choose k such that $mk > n + \frac{N}{2}$. Then by embedding theorem for the Sobolev's spaces $H^{km}(\Omega) \subset C^n(\Omega)$ (space of n times continuous differentiable functions) obtain that $E_\lambda \varphi \to \varphi$ in topology of the space $C^n(K)$, where K is arbitrary compact from Ω. From this, due to arbitrariness of the number n obtain that $E_\lambda \varphi \to \varphi$ in topology of the space $\mathrm{E}(\Omega)$. Thus we proved following

Theorem 2:

Spectral expansions of any distribution with compact support (3) converges in weak topology.

Fourier-Laplace Series On Sphere:

Denote by S^N N-dimensional unit sphere:

$$S^N = \left\{ x = (x_1, x_2, \ldots, x_{N+1}) \in R^{N+1} : \sum_{j=1}^{N+1} x_j^2 = 1 \right\}$$

Let $x, y \in S^N$. By $\gamma = \gamma(x, y)$ denote a geodesic distance between these two points of the sphere S^N. In fact that γ is value of the angle between x and y. We consider as Hilbert space H a space $L_2(S^N)$. A set of smooth functions $\{\varphi_{k,j}(x) = Y_k^j(x)\}_{j=1}^{a_k}$, $a_k = N_k - N_{k-2}$, $N_k = \frac{(N+k)!}{N!k!}$, $k = 1,2,3,\ldots,$ of spherical harmonics is an orthonormal system of functions from H. Moreover, a system $\{\varphi_{k,j}(x) = Y_k^j(x)\}_{k\, j=1}^{a}$ is an orthonormal basis in the space of spherical harmonics degree k.

For any distribution f from $\mathcal{E}'(S^N)$ define its coefficients by basis $\varphi_{k,j}(x)$ as $f_{k,j} = < f, \varphi_{k,j} >$. Then partial sums $E_n f(x)$ of its expansions by basis $\varphi_{k,j}(x)$ can be written as an action of the distribution f on a function

$$\Theta(x, y, n) = \sum_{k=0}^{n} \left(k + \frac{N-1}{2} \right)^{\frac{N-1}{2}} P_{\frac{N-1}{2}}^{k}(\cos\gamma),$$

where $P_\nu^k(t)$ is the Gegenbauer polynomials (the Legendre polynomials for $\nu = \frac{1}{2}$).

Note that $E_n \varphi(x) \to \varphi(x)$ when $n \to \infty$ in topology of the space $E(S^N)$. Then from the theorem above it follows

Theorem 3:

Expansions by spherical functions of any distribution from $E'(S^N)$ converges in the weak topology.

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