Australian Journal of Basic and Applied Sciences, 6(2): 34-48, 2012
ISSN 1991-8178

On A Certain Thermoelastic Problem Of Temperature And Thermal Stresses In A Thick Circular Plate

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Abstract: This paper is concerned with the determination of unknown temperature, displacement and thermal stresses on the upper surface of a thick circular plate subjected to an interior heat flux is known under an unsteady-state temperature field. The lower surface is kept at zero temperature and the fixed circular edge thermally insulated. The governing heat conduction equation has been solved by using the Laplace transform technique. The results are obtained in series form in terms of Bessel's functions and these have been computed numerically and illustrated graphically.

Key words: Unsteady state, Thermoelastic problem, Thermal stresses, Thick Circular plate, Michell's function.

INTRODUCTION

Circular plates are one of the most widely used structural elements in various engineering applications such as the pavements of highways and airports, building walls and bridge decks and so on. In most cases, the plates have to carry various loads. Therefore, a thorough understanding of their mechanics characteristics is essential for designers. Although most of the plates in applications have a constant thickness, the variable thickness plates have been also receiving an increasing attention from designers and researches. The use of variable thickness in plate design can reduce structure weight and maximize the material potential, especially for the aerospace applications.

The direct problems of the thermoelasticity in a thin circular plate have been considered by Nowacki (1957) and Roy Choudhari (1973). Wankhede (1982) has determined the quasi-static thermal stresses in a circular plate subjected to arbitrary temperature on the upper face with the lower face at zero temperature and the fixed circular edge thermally insulated. Grysa *et al*. (1982) investigated an inverse one-dimensional transient thermoelastic problem and obtained the temperature and heat flux on the surface of an isotropic infinite slab. Toshiaki (1982) has determined the thermal stresses in a non-homogeneous thick plate under steady distribution of temperature. Noda *et al*. (1989) discussed an analytical method for an inverse problem of three-dimensional transient thermoelasticity in a transversely isotropic solid. Rogers and Spencer (1989) extended the equivalent two-dimensional plate bending theory to include thermal effects and applied it to symmetric and antisymmetric temperature distributions in a homogeneous and isotropic plate, as well as to a laminated plate of distinct isotropic and homogeneous layers. Tanigawa *et al*. (1996; 1997) have studied the theoretical analysis of two-dimensional thermoelastoplastic deformation of plate subject to partially distributed heat supply. Yongzhi Xu (1999) has determined an inverse problem for quasi-static, linearized, thermoelastic system on the unit disk. Ashida *et al*. (2002) discussed the inverse transient thermoelastic problem for composite circular disc. Khobragade *et al*. (2003) solved an inverse unsteady-state thermoelastic problem of a thin circular plate in Marchi-Fasulo transform domain. Kang (2003) have studied the three-dimensional vibration analysis of thick, circular and annular plates with nonlinear thickness variation. Ma and Wang (2003) investigated axisymmetric large deflection analysis of a functionally graded circular plate. Ootao and Tanigawa (2004) have studied the theoretical treatment of a transient thermoelastic problem involving an FG thick strip due to a nonuniform heat supply in the width direction.

Qian and Batra (2004) solved the transient thermoelastic deformations of a thick functionally graded plate with edges held at a uniform temperature and either simply supported or clamped. Sharma *et al*. (2004) have studied the behavior of thermoelastic thick plate under lateral loads. Khobragade *et al*. (2005) solved thermal deformation in a thin circular plate due to a partially distributed heat supply. Gaikwad M.N. (2005) solved the inverse problem of thermoelasticity in a thin isotropic circular plate by determining the unknown temperature gradient, temperature distribution and the thermal deflection on the edge of the circular plate. Li *et al*. (2006) studied the pure bending problem of simply supported transversely isotropic circular plates with elastic compliance coefficients being arbitrary functions of the thickness coordinate. Also Imrak and Gerdemeli (2007)

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analysis the deflections of a rectangular fixed thin plates under uniformly distributed loads. Kulkarni *et al*. (2007) has determined the quasi-static thermal stresses in a thick circular plate subjected to arbitrary initial temperature on the upper surface. Recently, Ghadle *et al*. (2011) solved an inverse quasi-static thermoelastic problem in a thick circular plate.

In this article, we analyzed inverse thermoelastic problem of temperature and thermal stresses in a thick circular plate. Determined the expressions for unknown temperature, displacement and thermal stresses on the upper surface $(z = h)$ of a thick circular plate subjected to an interior heat flux $(-Q_0 f(r)/\lambda)$ are known under an unsteady-state temperature field. The lower surface $(z = -h)$ is kept at zero temperature and the fixed circular edge $(r = a)$ thermally insulated. The governing heat conduction equation has been solved by using the Laplace transform technique. The results are obtained in series form in terms of Bessel's functions and these have been computed numerically and illustrated graphically.

It is believed that, this particular problem has not been previously considered. The results presented here will be more useful in engineering problem particularly in the determination of the state of strain in thick circular plate constituting foundations of containers for hot gases or liquids, in the foundations for furnaces etc.

Statement Of The Problem:

We consider a thick circular plate of radius a and thickness *2h* occupying space *D*: $0 \le r \le a, -h \le z \le h$. Initially the plate is at zero temperature. Let the plate be subjected to an interior heat flux $(-Q_0 f(r)/\lambda)$ is known within region $-h \le \xi \le h$. The lower surface $(z = -h)$ is kept at zero temperature and the fixed circular edge $(r = a)$ thermally insulated. Assume that the boundary of the circular plate is free from traction.

Under these more realistic prescribed conditions, the unknown temperature $g(r)$ which is at the upper surface $(z = h)$ of the plate and the quasi-static thermal stresses due to unknown temperature $g(r)$ need to be determined.

The differential equation governing the displacement potential function $\phi(r, z, t)$ is given in (Noda, N., 2003) as,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = K\tau \tag{1}$$

where K is the restraint coefficient and temperature change $\tau = T - T_i$, where T_i is initial temperature. The displacement function ϕ is known as Goodier's thermoelastic displacement potential.

The differential equation governing the temperature function $T(r, z, t)$ is given by,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2}$$

where k is the thermal diffusivity of the material of the plate,
subject to the initial condition

$$T = 0 \qquad \text{at } t = 0, \tag{3}$$

subject to the boundary condition's

$$\frac{\partial T}{\partial r} = 0, \qquad \text{at } r = a,\ -h \le z \le h \tag{4}$$

$$\frac{\partial T}{\partial z} = 0, \qquad \text{at } z = -h, -h \le z \le h \tag{5}$$

$$\frac{\partial T}{\partial z} = g(r) \text{ (Unknown)}, \qquad \text{at } z = h,\ 0 \le r \le a \tag{6}$$

$$\frac{\partial T}{\partial z} = \frac{-Q_0}{\lambda} f(r) \text{ (Known)}, \qquad \text{at } z = \xi, -h \le \xi \le h \tag{7}$$

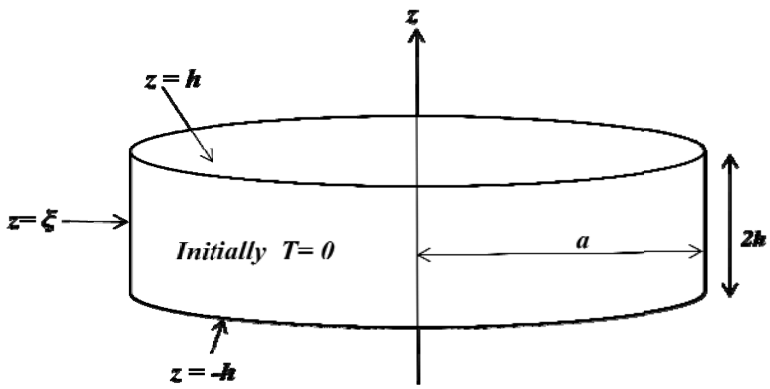

Fig. 1: the geometry of the problem.

The displacement function in the cylindrical coordinate system are represented by the Michell's function defined in (Noda, N., 2003) as,

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \tag{8}$$

$$u_z = \frac{\partial \phi}{\partial r} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{9}$$

The Michell's function must satisfy

$$\nabla^2 \nabla^2 M = 0 \tag{10}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{11}$$

The components of the stresses are represented by the thermoelastic displacements potential and Michell's function as,

$$\sigma_{rr} = 2G\left[\frac{\partial^2 \phi}{\partial r^2} - K\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{12}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial \phi}{\partial r} - K\tau + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{13}$$

$$\sigma_{zz} = 2G\left[\frac{\partial^2 \phi}{\partial z^2} - K\tau + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{14}$$

and

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial z}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{15}$$

The boundary conditions on the traction free surfaces of an circular plate are,

$$\sigma_{zz} = \sigma_{rz} = 0 \text{ , at } z = \pm h \tag{16}$$

Equations (3.1) to (3.15) constitute the mathematical formulation of the problem under consideration.

Solution Of The Problem:

Taking the Laplace transform defined in (Sneddon I.N., 1972) to the Eqs. (2), (4), (5) and (7) with respect to t and using (3), one obtains

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} + \frac{\partial^2 \overline{T}}{\partial z^2} = \frac{p}{k}\overline{T} \tag{17}$$

with boundary conditions

$$\frac{\partial \overline{T}}{\partial r} = 0 , \qquad \text{at } r = a , \; -h \le z \le h \tag{18}$$

$$\frac{\partial \overline{T}}{\partial z} = 0 , \qquad \text{at } z = -h , 0 \le r \le a \tag{19}$$

$$\frac{\partial \overline{T}}{\partial z} = \frac{-Q_0}{\lambda} f(r), \qquad \text{at } z = \xi , -z \le \xi \le z \tag{20}$$

where p is the Laplace transform parameter.

Now applying method of separation of variable to solve equation (17), one obtains

$$\overline{T}(r,z,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)\cosh\left[\gamma_n (z+h)\right] \tag{21}$$

and it satisfying equation (19) and (20),

where

$$\gamma_n = \left(\lambda_n^{\,2} + \frac{p}{k}\right)^{1/2} \qquad n = 1, 2, ...\infty$$

(22)

and $\lambda_1, \lambda_2, ...$ are the roots of the transcendental equation

$$J_1(\lambda a) = 0 \tag{23}$$

with $J_n(x)$ is Bessel function of the first kind of order n and A_n is constant. The constant A_n can be obtained by using equations (18) and (21),

$$\frac{-Q_0}{\lambda p} f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)\gamma_n \sinh\left[\gamma_n (\xi + h)\right]$$

By theory of Bessel's function

$$\int_0^a \left(\frac{-Q_0}{\lambda p} f(r)\right) r\, J_0(\lambda_n r)\, dr = \int_0^a A_n J_0^2(\lambda_n r)\gamma_n \sinh\left[\gamma_n(\xi+h)\right] dr$$

Using

$$\int_0^a r\, J_0^2(\lambda_n r)\, dr = \left(\frac{a^2}{2}\right) J_0^2(\lambda_n a)$$

One obtains

$$A_n = \left(\frac{-2Q_0 \overline{f}(\lambda_n)}{a^2 \lambda p}\right) \sum_{n=1}^{\infty} \left[\frac{J_1(\lambda_n r)}{\lambda_n \gamma_n \left[J_0(\lambda_n a)\right]^2 \sinh\left[\gamma_n(\xi+h)\right]}\right] \tag{24}$$

where

$$\overline{f}(\lambda_n) = \int_0^a r\, J_0(\lambda_n r)\, f(r)\, dr \tag{25}$$

Using Eqs. (23) and (24) in Eq. (21), one obtains

$$\overline{T}(r,z,p)=\left(\frac{-2Q_0}{a^2 p}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left[\frac{\cosh\left[\left(\lambda_n{}^2+\frac{p}{k}\right)^{1/2}(z+h)\right]}{p\left(\lambda_n{}^2+\frac{p}{k}\right)^{1/2}\sinh\left[\left(\lambda_n{}^2+\frac{p}{k}\right)^{1/2}(\xi+h)\right]}\right] \tag{26}$$

Finally applying the inverse Laplace transform defined in (Sneddon I.N., 1972) to the equation (26), one obtains the expression for temperature

$$T(r,z,t)=\left(\frac{2Q_0}{a^2 p}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{-\cosh\left[\lambda_n(z+h)\right]}{\lambda_n\sinh\left[\lambda_n(\xi+h)\right]}\right]+\frac{e^{-k\lambda_n^2 t}}{\lambda_n^2(\xi+h)}\right.$$
$$\left.-\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)}\right]\right\} \tag{27}$$

Since $T_i=0$, the temperature change is $\tau=T-T_i=T.$ (28)

Unknown Temperature:

The unknown temperature $g(r)$ can be obtained by substituting $z=h$ into Eq. (27), one obtains

$$g(r)=\left(\frac{2Q_0}{a^2 p}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{-\sinh\left[2\lambda_n h\right]}{\sinh\left[\lambda_n(\xi+h)\right]}\right]\right.$$
$$\left.+\frac{2\pi}{(\xi+h)^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}m\sin\left[\left(\frac{m\pi}{(\xi+h)}\right)(2h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)}\right]\right\} \tag{29}$$

Michell's Function:

A suitable form of M satisfying Eq. (10) is given by

$$M=\left(\frac{2Q_0 K}{a^2 p}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\times\left\{C_{mn}\sinh\left[\lambda_n(z+h)\right]\rangle+D_{mn}\lambda_n(z+h)\cosh\left[\lambda_n(z+h)\right]\right\} \tag{30}$$

where C_{mn} and D_{mn} are arbitrary constants, which can be determined finally by using conditions (16).

Displacement Potential Function:

To obtain displacement potential function $\phi(r,z,t)$ using equation (27) and (28) in equation (1), one obtains

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{2Q_0 K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{-\cosh\left[\lambda_n(z+h)\right]}{\lambda_n\sinh\left[\lambda_n(\xi+h)\right]}\right]+\frac{e^{-k\lambda_n^2 t}}{\lambda_n^2(\xi+h)}\right.$$
$$\left.-\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)}\right]\right\} \tag{31}$$

Considering the first term on R.H.S. of equation (31) as,

$$\frac{\partial^2\phi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_1}{\partial r}+\frac{\partial^2\phi_1}{\partial z^2}=\left(\frac{2Q_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left[\frac{-\cosh\left[\lambda_n(z+h)\right]}{\lambda_n\sinh\left[\lambda_n(\xi+h)\right]}\right] \tag{32}$$

To solve equation (32) assume ϕ_1 which satisfies equation

$$\phi_1=\sum_{n=1}^{\infty}\left\{E_nJ_0(\lambda_n r)\left[-(z+h)\sinh\left[\lambda_n(z+h)\right]\right]\right\} \tag{33}$$

and using (33) in (32), one obtains one obtains

$$E_n=\left(\frac{Q_0K\overline{f}(\lambda_n)}{2a^2\lambda\left[J_0(\lambda_n a)\right]^2\lambda_n^2\sinh\left[\lambda_n(\xi+h)\right]}\right)$$

Thus the Eq. (33) becomes

$$\phi_1=\left(\frac{Q_0K}{\lambda}\right)\sum_{n=1}^{\infty}\left(\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{a^2\left[J_0(\lambda_n a)\right]^2}\right)\left[\frac{-(z+h)\sinh\left[\lambda_n(z+h)\right]}{2\lambda_n^2\sinh\left[\lambda_n(\xi+h)\right]}\right] \tag{34}$$

Now considering second and third term on R.H.S. of equation (31) as,

$$\frac{\partial^2\phi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_2}{\partial r}+\frac{\partial^2\phi_2}{\partial z^2}=\left(\frac{2Q_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\frac{e^{-k\lambda_n^2t}}{\lambda_n^2(\xi+h)}-\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)}\right]\right\} \tag{35}$$

To solve equation (35) using

$$\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\approx\frac{1}{k}\frac{\partial T}{\partial t} \tag{36}$$

On integrating w.r.t. *t*, one obtains

$$\phi_2=\left(\frac{Q_0K}{\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\sqrt{N}}\right]\left\{-\frac{e^{-k\lambda_n^2t}}{\lambda_n^4(\xi+h)}-\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right\} \tag{37}$$

Finally $\phi=\phi_1+\phi_2$, one obtains

$$\phi=\left(\frac{Q_0K}{\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\sqrt{N}}\right]\left\{\left[\frac{-(z+h)\sinh\left[\lambda_n(z+h)\right]}{2\lambda_n^2\sinh\left[\lambda_n(\xi+h)\right]}\right]-\frac{e^{-k\lambda_n^2t}}{\lambda_n^4(\xi+h)}-\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right\} \tag{38}$$

Displacement Components And Thermal Stresses:

Now using Eqs. (28), (30) and (38) in Eqs. (8), (9) and (12)-(15), one obtains the expressions for displacements and stresses respectively as

$$u_r=\left(\frac{2Q_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_1(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{(z+h)\sinh\left[\lambda_n(z+h)\right]}{2\lambda_n\sinh\left[\lambda_n(\xi+h)\right]}\right]-\frac{e^{-k\lambda_n^2t}}{\lambda_n^2(\xi+h)}+\frac{2\lambda_n}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m}\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right.$$
$$\left.+\lambda_n^2C_{mn}\cosh\left[\lambda_n(z+h)\right]+D_{mn}\lambda_n^2\left[\cosh\left[\lambda_n(z+h)\right]+\lambda_n(z+h)\sinh\left[\lambda_n(z+h)\right]\right]\right\} \tag{39}$$

$$u_z=\left(\frac{2Q_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{-\left[\frac{\sinh[\lambda_n(z+h)]+\lambda_n(z+h)\cosh[\lambda_n(z+h)]}{2\lambda_n\sinh[\lambda_n(\xi+h)]}\right]+\frac{2\pi}{(\xi+h)^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^m m\sin\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right.$$
$$\left.-\lambda_n^2C_{mn}\sinh[\lambda_n(z+h)]+D_{mn}\lambda_n^2\left[2(1-2\nu)\sinh[\lambda_n(z+h)]-\lambda_n(z+h)\cosh[\lambda_n(z+h)]\right]\right\} \tag{40}$$

$$\sigma_{rr}=\left(\frac{4GQ_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_1(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\left(\frac{J_1(\lambda_n r)}{r}-\lambda_nJ_0(\lambda_n r)\right)\frac{-(z+h)\sinh[\lambda_n(z+h)]}{2\lambda_n\sinh[\lambda_n(\xi+h)]}+\frac{J_0(\lambda_n r)\cosh[\lambda_n(z+h)]}{\lambda_n\sinh[\lambda_n(\xi+h)]}\right]-\left(\frac{J_1(\lambda_n r)}{r}\right)\left[\frac{e^{-k\lambda_n^2t}}{\lambda_n^3(\xi+h)}\right]\right.$$
$$-\left[\frac{m^2\pi^2}{(\xi+h)^2}J_0(\lambda_n r)-\frac{\lambda_n}{r}J_1(\lambda_n r)\right]\times\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^m\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]$$
$$\left.+\lambda_n^2C_{mn}\left[\lambda_nJ_0(\lambda_n r)-\frac{J_1(\lambda_n r)}{r}\right]\cosh[\lambda_n(z+h)]+D_{mn}\lambda_n^2\left[\left(-\lambda_nJ_0(\lambda_n r)+\frac{J_1(\lambda_n r)}{r}\right)\cosh[\lambda_n(z+h)]+\lambda_n(z+h)\sinh[\lambda_n(z+h)]+2\nu J_0(\lambda_n r)\cosh[\lambda_n(z+h)]\right]\right\} \tag{41}$$

$$\sigma_{\theta\theta}=\left(\frac{4GQ_0K}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_1(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{J_0(\lambda_n r)\cosh[\lambda_n(z+h)]}{\lambda_n\sinh[\lambda_n(\xi+h)]}+\left(\frac{J_1(\lambda_n r)}{r}\right)\frac{(z+h)\sinh[\lambda_n(z+h)]}{2\lambda_n\sinh[\lambda_n(\xi+h)]}\right]+\left(-\lambda_nJ_0(\lambda_n r)+\frac{J_1(\lambda_n r)}{r}\right)\left[\frac{e^{-k\lambda_n^2t}}{\lambda_n^3(\xi+h)}\right]\right.$$
$$+\left[\left(\frac{m^2\pi^2}{(\xi+h)^2}+\lambda_n^2\right)J_0(\lambda_n r)+\lambda_n\frac{J_1(\lambda_n r)}{r}\right]\times\frac{2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^m\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]$$
$$\left.+\lambda_n^2C_{mn}\left(\frac{J_1(\lambda_n r)}{r}\right)\cosh[\lambda_n(z+h)]+D_{mn}\lambda_n^2\left[2\nu\lambda_nJ_0(\lambda_n r)\cosh[\lambda_n(z+h)]+\lambda_n\left(\frac{J_1(\lambda_n r)}{r}\right)(z+h)\sinh[\lambda_n(z+h)]+\left(\frac{J_1(\lambda_n r)}{r}\right)\cosh[\lambda_n(z+h)]\right]\right\} \tag{42}$$

$$\sigma_{zz}=\left(\frac{4GKQ_0}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_0(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{-(z+h)\sinh[\lambda_n(z+h)]}{2\sinh[\lambda_n(\xi+h)]}\right]-\frac{e^{-k\lambda_n^2t}}{\lambda_n^2(\xi+h)}+\frac{2\lambda_n^2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^m\cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right.$$
$$\left.-\lambda_n^2C_{mn}\cosh[\lambda_n(z+h)]+D_{mn}\lambda_n^3\left[(1-2\nu)\cosh[\lambda_n(z+h)]-\lambda_n(z+h)\sinh[\lambda_n(z+h)]\right]\right\} \tag{43}$$

$$\sigma_{rz}=\left(\frac{4GKQ_0}{a^2\lambda}\right)\sum_{n=1}^{\infty}\left[\frac{\overline{f}(\lambda_n)J_1(\lambda_n r)}{\lambda_n\left[J_0(\lambda_n a)\right]^2}\right]\left\{\left[\frac{\sinh[\lambda_n(z+h)]+\lambda_n(z+h)\cosh[\lambda_n(z+h)]}{2\lambda_n\sinh[\lambda_n(\xi+h)]}\right]-\frac{2\pi\lambda_n}{(\xi+h)^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^m m\sin\left[\left(\frac{m\pi}{(\xi+h)}\right)(z+h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]\right.$$
$$\left.+\lambda_n^2C_{mn}\sinh[\lambda_n(z+h)]+D_{mn}\lambda_n^2\left[2\nu\sinh[\lambda_n(z+h)]+\lambda_n(z+h)\cosh[\lambda_n(z+h)]\right]\right\} \tag{44}$$

Now in order to satisfy Eq. (16), solving Eqs. (37) and (38) for C_{mn} and D_{mn}, one obtains

$$C_{mn}=\sum_{n=1}^{\infty}\left\{\frac{2\nu-1}{\lambda_n^4\sinh[\lambda_n(\xi+h)]}-\left[\frac{F_{mn}\left[2\nu\sinh[2\lambda_nh]+2\lambda_nh\cosh[2\lambda_nh]\right]}{\lambda_n^4\left[\sinh[2\lambda_nh]\cosh[2\lambda_nh]+2\lambda_nh\right]}\right]+\left[\frac{G_{mn}\left[(1-2\nu)\cosh[2\lambda_nh]-2\lambda_nh\sinh[2\lambda_nh]\right]}{\lambda_n^4\left[\sinh[2\lambda_nh]\cosh[2\lambda_nh]+2\lambda_nh\right]}\right]\right\} \tag{45}$$

$$D_{mn}=\sum_{n=1}^{\infty}\left\{\frac{-1}{\lambda_n^4\sinh[\lambda_n(\xi+h)]}-\left[\frac{F_{mn}\sinh[2\lambda_nh]-G_{mn}\cosh[2\lambda_nh]}{\lambda_n^4\left[\sinh[2\lambda_nh]\cosh[2\lambda_nh]+2\lambda_nh\right]}\right]\right\} \tag{46}$$

where

$$F_{mn} = -\frac{e^{-k\lambda_n^2 t}}{\lambda_n^2(\xi+h)} - \frac{2\lambda_n^2}{(\xi+h)}\sum_{m=1}^{\infty}\left[\frac{(-1)^m \cos\left[\left(\frac{m\pi}{(\xi+h)}\right)(2h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]$$

$$G_{mn} = \frac{2\pi\lambda_n}{(\xi+h)^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^m m\sin\left[\left(\frac{m\pi}{(\xi+h)}\right)(2h)\right]e^{-k\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)t}}{\left(\lambda_n^2+\frac{m^2\pi^2}{(\xi+h)^2}\right)^2}\right]$$

Special Case And Numerical Calculations:

Setting

$f(r) = T_0\delta(r-b), \qquad (a > b)$

in equation (25), where T_0 is constant and $b < a$.

$\overline{f}(\lambda_n) = bJ_0(\lambda_n b)$.

The numerical calculation has been carried out for Copper(pure) plate with the parameters *a = 2m, b = 1m, h= 0.4m,* ξ *= 0.2m*, thermal diffusivity $k = 112.34\times10^{-6}$ (m^2s^{-1}), Poisson ratio ν = 0.35 with λ_1 = 3.8317, λ_2 = 7.0156, λ_3 = 10.1735, λ_4 = 13.3237, λ_5 = 16.470, λ_6 = 19.6159, λ_7 = 22.7601, λ_8 = 25.9037, λ_9 = 29.0468, λ_{10} = 32.18 being the positive roots of transcendental equation $J_1(\lambda a) = 0$ defined in (Ozisik, N.M., 1968).

For convenience, setting

$$\alpha = \left(\frac{2Q_0}{a^2\lambda}\right) \qquad \beta = \left(\frac{2Q_0K}{a^2\lambda}\right) \qquad \gamma = \left(\frac{4GQ_0K}{a^2\lambda}\right)$$

in the expressions (29) and (39) to (44).

In order to examine the influence of unsteady state temperature field on the thick plate, one performed the numerical calculations for the different radii *r* = 0, 0.5, 1, 1.5, 2*m* and for the different time *t* = 1, 2, 3, 4, 5 *sec* are shown in the figures with the help of computational mathematical software Mathcad-2007, and the graphs are plotted with the help of Excel (MS Office-2007).

Discussion:

In this study, we develop the analysis for the temperature field by introducing the methods of the Laplace transforms and determined the expressions for unknown temperature, displacement and thermal stresses on the upper surface of a thick circular plate subjected to an interior heat flux is known under an unsteady-state field.

As a special case mathematical model is constructed for

$f(r) = T_0\delta(r-b), \qquad (a > b)$

and performed numerical calculations.

From figure 2, represents the graph of unknown temperature $g(r)$ versus radius r for different time parameters t = 1, 2, 3, 4, 5. It is clear that for fixed r and with the increase of time t the unknown temperature decreases and at last it reaches steady state at r =2, which is physically plausible. **From figure 3,** depicts $g(r)$ versus time t for different radii r = 0, 0.5, 1, 1.5, 2 *m*. It is clear from the graph that $g(r)$ goes on increasing as the time increases for the radii of the plate.

From figure 4, represents the graph of radial displacement function u_r . It is seen that u_r vanishes at r = 2. It is clear that for fixed r and with the increase of time t , u_r increases and it decreases with the increase of r , which is physically plausible. **From figure 5,** it is observed that u_r increases for different radii and it remains constant within region $2 \le t \le 5$

From figure 6, shows the variation of axial displacement u_z versus radius r for different time. It is clear that u_z increases with the time within the circular region $0 \le r \le 1$ and decreases within annular region $1 \le r \le 2$. It develops the compressive stresses within circular region $0 \le r \le 1$ and tensile stresses in annular region $1 \le r \le 2$. **From figure 7,** depicts the graph of u_z versus t for different radii. Here we observe that u_z increases with the increase of time t.

From figure 8, represents the graph of radial stress σ_{rr}, it is clear that σ_{rr} increases with the time within the circular region $0 \le r \le 1.125$ and decreases within annular region $1.125 \le r \le 2$. **From figure 9,** it is observe that σ_{rr} increases with the increase of time t.

From figure 10, shows the variation of angular stress $\sigma_{\theta\theta}$, it is clear that $\sigma_{\theta\theta}$ increases with the time increases within the circular region $0 \le r \le 0.8$ and decreases with the time increases within annular region $0.8 \le r \le 2$. **From figure 11,** depicts the graph of $\sigma_{\theta\theta}$ it is observe that $\sigma_{\theta\theta}$ decreases with the increase of r and remains constant within region $3 \le r \le 5$.

From figure 12, represents the graph of axial stress σ_{zz}, it is seen that σ_{zz} increase with the progress of time and approaches the maximum values at steady state. **From figure 13,** depicts the graph of σ_{zz} it is observe that σ_{zz} decreases with the increase of r and remains constant within region $3 \le r \le 5$.

From figure 14, represents the graph of resultant stress σ_{rz}, vanishes at $r = 2$. It is clear that for fixed r and with the increase of time t, σ_{rz} increases and it decreases with the increase of r, which is physically plausible. From **figure 15,** depicts the graph of σ_{rz}, it is observed that σ_{rz} increases for different radii and it remains constant within region $2 \le t \le 5$.

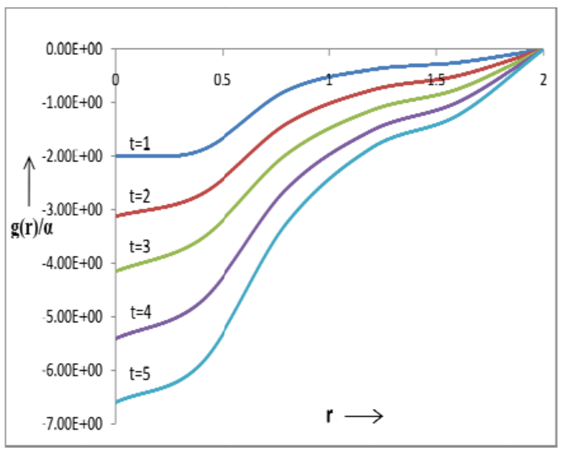

Fig. 2: The unknown temperature $g(r)/\alpha$ vs. radius (for different times).

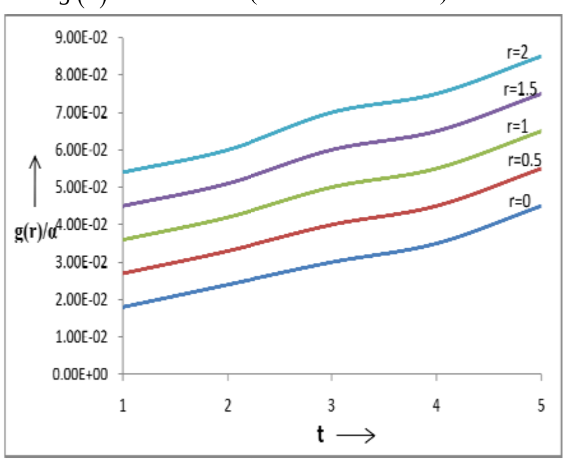

Fig. 3: The unknown temperature $g(r)/\alpha$ vs. time (for different radii).

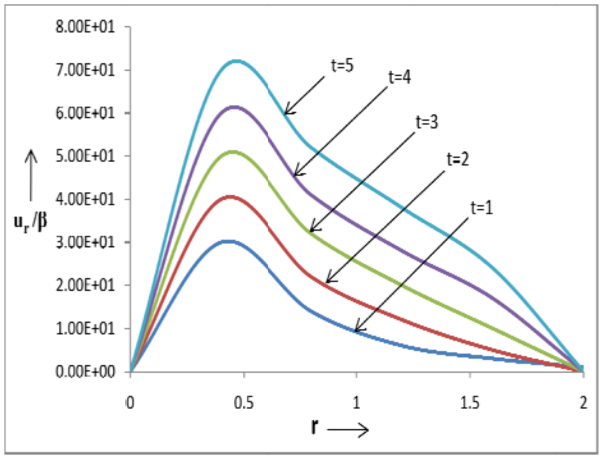

Fig. 4: The radial displacements u_r / β vs. radius (for different times).

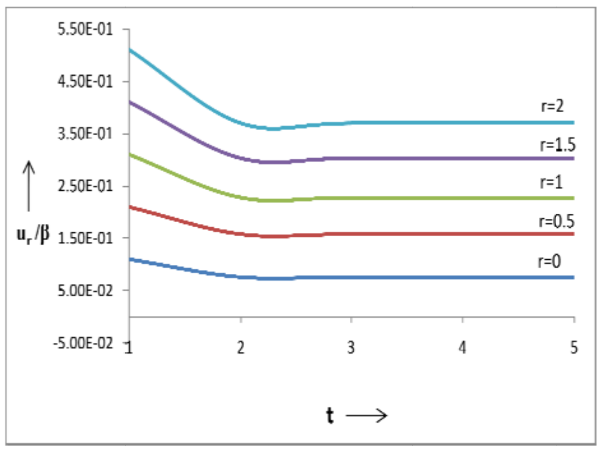

Fig. 5: The radial displacements vs. time (for different radii).

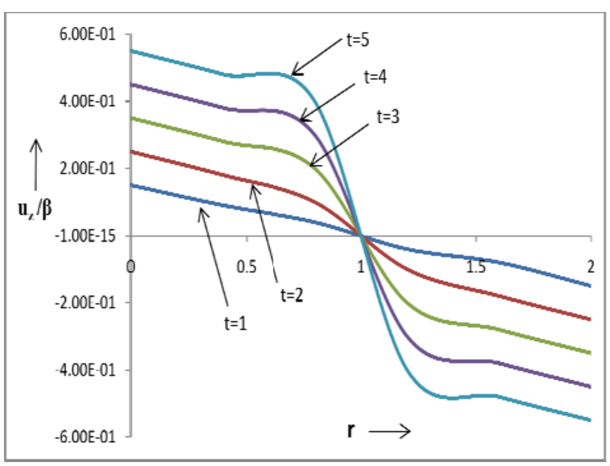

Fig. 6: The radial displacements u_z / β vs. radius (for different times).

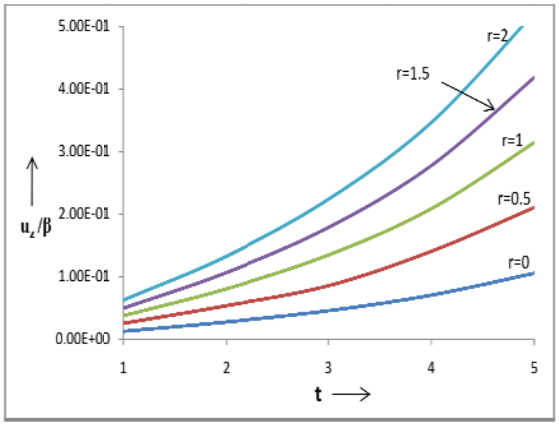

Fig. 7: The radial displacements u_z / β vs. time (for different radii).

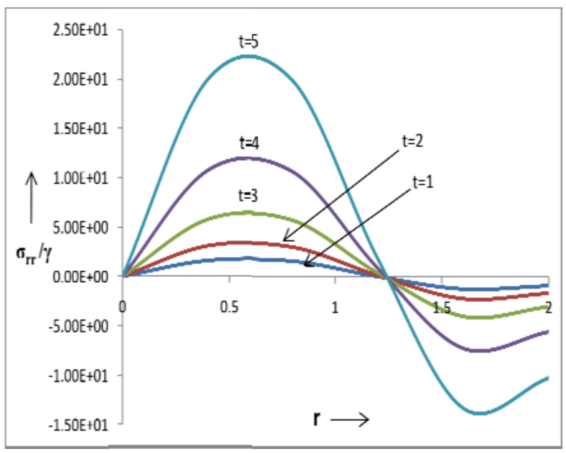

Fig. 8: The radial stress σ_{rr} / γ vs. radius (for different times).

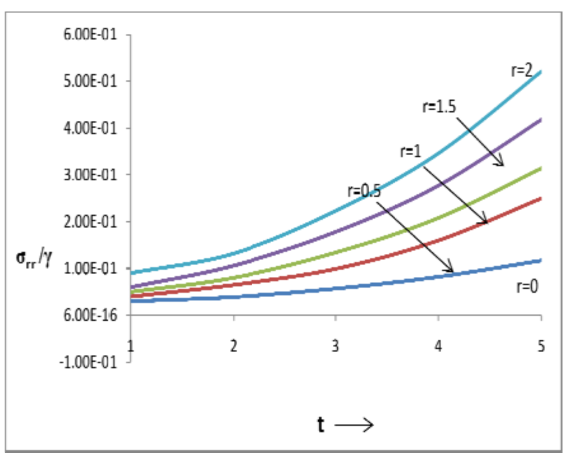

Fig. 9: The radial stress σ_{rr} / γ vs. time (for different radii).

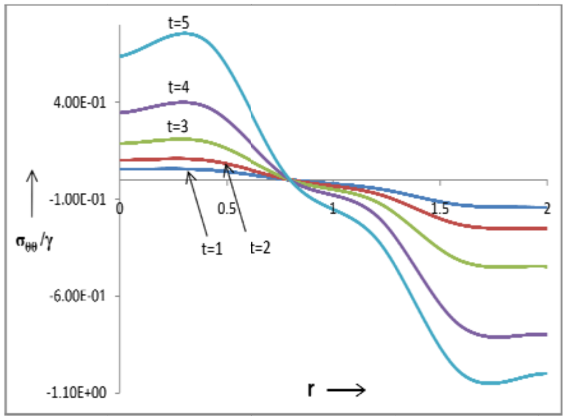

Fig. 10: The angular stress $\sigma_{\theta\theta} / \gamma$ vs. radius (for different times).

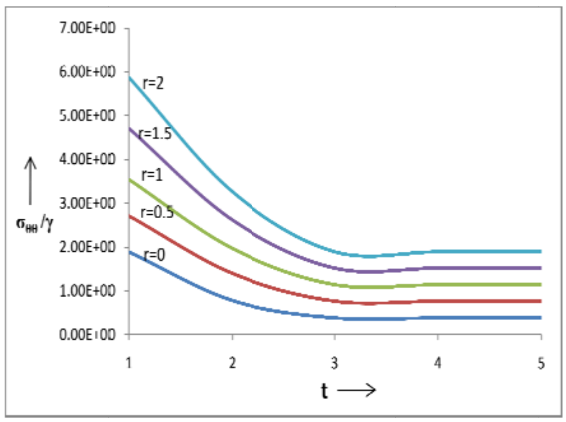

Fig. 11: The angular stress $\sigma_{\theta\theta} / \gamma$ vs. time (for different radii).

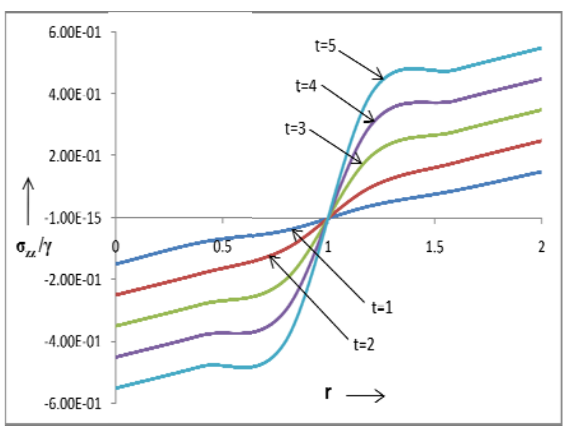

Fig. 12: The axial stress σ_{zz} / γ / vs. radius (for different times).

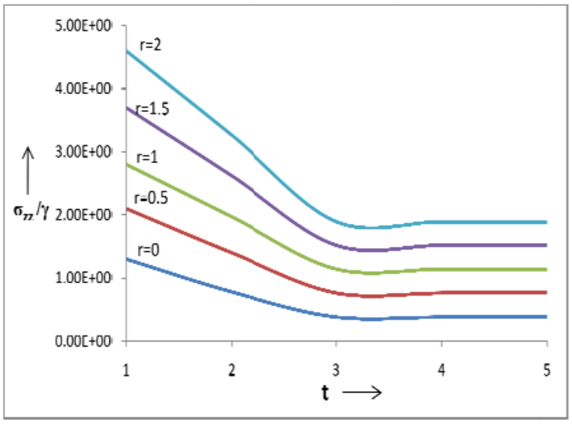

Fig. 13: The axial stress $\sigma_{zz} / \gamma /$ vs. time (for different radii).

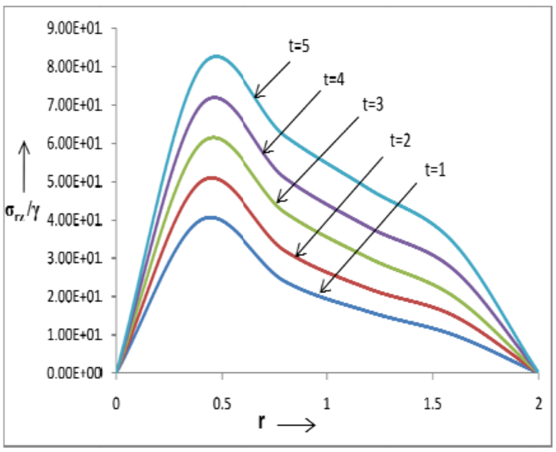

Fig. 14: The resultant stress $\sigma_{rz} / \gamma /$ vs. radius (for different times).

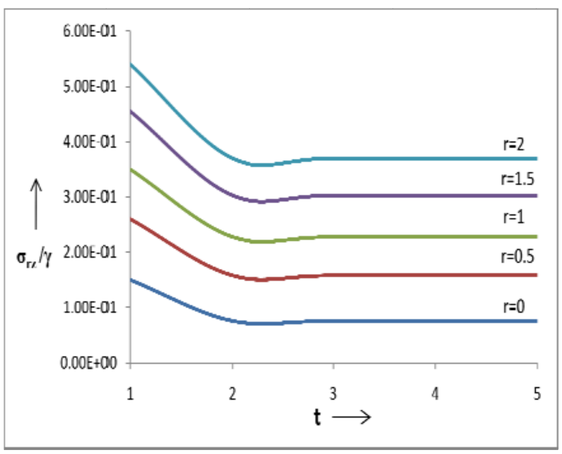

Fig. 15: The resultant stress $\sigma_{rz} / \gamma /$ vs. time (for different radii).

Concluding Remarks:

On a certain thermoelastic problem of temperature and thermal stresses in a thick circular plate is presented in this paper. The governing heat conduction equation has been solved by using the Laplace transform technique. The results for unknown temperature, displacement and thermal stresses are obtained in series form in terms of Bessel's functions and these have been computed numerically and illustrated graphically.

We can summarize that, the stress components and displacement occurs near heated region. From the figures of radial and axial displacements, it can observe that the radial displacement occur away from the center ($r = 0$). From the figures of axial displacement is in the upward direction at the center and downward direction near the circular boundary of the plate. The Radial stress component σ_{rr} develops the compressive stresses within circular region $0 \le r \le 1.125$ near heat source and tensile stresses within annular region $1.125 \le r \le 2$ out of heat source in radial direction, where as angular stress $\sigma_{\theta\theta}$ develops the compressive stresses within circular region $0 \le r \le 0.8$ near heat source and tensile stresses in annular region $0.8 \le r \le 2$ out of heat source in radial direction. With the temperature increases the circular plate will tend to expand in radial direction. We concluded that, due to unknown temperature arbitrary heat applied on the upper surface under unsteady state temperature field, the circular plate expands in axial direction and bends concavely at the center.

The results obtained here are more useful in engineering problems particularly in the determination of state of strain in thick circular plate. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (29) and (39)-(44).

ACKNOWLEDGEMENT

The author's very grateful thanks to the anonymous referee for their kind help and active guidance in the preparation of this revised paper.

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