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Knots and Colorability

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Abstract: We have established that tricolourability would be a way of distinguishing some of knots (links) by showing that tricolorability is an ambient isotopy invariant. We have extended the notion of tricolorability to colorability of knot(link) and have shown that colorability of knot (link) is also an ambient isotopy invariant. We have shown that no knot is colorable mod 2 but instead every link with more than one component is colorable mod 2. We have also established that bridge number of a knot is always one.

Key words: Reidemeister moves, ambient isotopy invariants, colourability. Mathematics Subject Classification 2000: 57M25(15) (27).

INTRODUCTION

Mathematicians were perplexed at the seemingly unending number of ways a knot could be shaped and turned. Consequently, these give rise to the central problem of knot theory i.e., whether two knots (links) are equivalent or not. This was the motivation for much of the recent work in knot theory, which is devoted to search for invariants of knots. The study of invariants underwent in a kind of phase transition, which has linked knot theory to chemistry, molecular chemistry, mathematical physics, particles physics, polymer physics, statistical mechanics, fluid mechanics, kinematics, C^*-algebra, conformal field theory, crystallography, cryptography, graph theory, computer systems and networks, etc. In the recent past, biologists and chemists studying genetics discovered an exciting link of knot theory with DNA (genetic material of all cells, containing coded information about cellular molecules and processes) and synthetic chemistry. DNA is just one application of knot theory, which presently is an area of intense mathematical activities worldwide.

The story of knot coloring started in 1956 by Ralph Fox (1963; 1970). He has delivered a beautiful lecture to undergraduate students at Haverford College. His lecture was so outstanding that it has infact changed the history of topology. His lecture was about colored knots. He has not introduced the concept by a usual approach that is homomorphisms from the knot group onto a dihedral group, but instead he introduced knot colorings by physically colouring arcs of knot diagrams red, blue, and green with some ruling.

In this paper We will establish that tricolourability would be a way of distinguishing some of knots(links) by showing that tricolorability is an ambient isotopy invariant. We will extend the notion of tricolorability to colorability of knot(link) and will show that colorability of knot (link) is also an ambient isotopy invariant. We will show that no knot is colorable mod 2 but instead every link with more than one component is colorable mod 2. We will also establish that bridge number of a knot is always one.

MATERIAL AND METHODS

Terminology, definition and concept about knot(link), projection, regions, under-crossing(over-crossing), Reidemeister moves etc. are the usual one (D Rolfsen. 1976; G. Burde and H. Zieschang, 2003; R.H.Crowell and R.H.Fox, 1963; W.B.R. Lickorish, 1976).

By the planar isotopy we mean the motion of the projection in the plane that preserves the graphical structure of the underlying universe. The pivotal moves in the theory of knots are the Reidemeister moves. We will view these moves as Reidemeister moves (K. Reidemeister, 1983) of type I, II, and III.

Graph theoretic versions of these moves have been discussed in detail (M. Azram, 1993). Two knots (links) in space can be deformed into each other (ambient isotopy) if and only if their projections can be transformed into one another by planar isotopy and the three Reidemeister moves. Two knots are equivalent (via Reidemeister moves) if and only if (any of) their projections differ by a finite sequence of Reidemeister moves (K. Reidemeister, 1983). Ambient isotopy and equivalence via Reidemeister moves is the same (G. Burde and H. Zieschang, 2003).

In the future, knots (links) will be confused with their class of projections with crossings indicated unless otherwise stated.

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RESULTS AND DISCUSSION

The results and discussion in this article are variational, diagrammatic and illustrative. R^*-move (Figure 1) is a well defined move via Reidemeister moves (M. Azram, 2003). Infact this is a generalized form of Reidemeister move of type II. A 2π-twist and/or a π-twist move (Figure 2) which are just consequence of R^*-move (M. Azram, 2003) will be pivotal moves in the discussion hereafter.

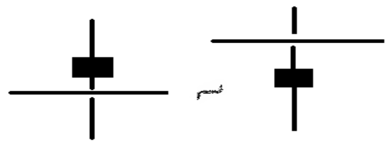


Fig. 1: (R^*-move).

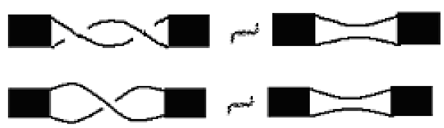


Fig. 2: (2π-twist and a π-twist moves).

Any of the R^*-move, 2π-twist or π-twist moves especially π-twist move can be used to remove an isthmus crossing (if any two of the four local regions at the crossing are parts of the same region in the whole diagram). Consequently;

Definition:

A bridge is a string between two under-crossings (over-crossings) with no under-crossing (over-crossing) in-between and at least one over-crossing (under-crossing).

Definition:

The bridge number of a knot K denoted as b (K) is the minimum number of bridges that occur ranging over all possible for that knot.

Proposition:

$$b(K) = 1 \Rightarrow K \text{ is } un - knot$$

Proof:

Assume K is not unknot. Assume K has the minimum number of crossings but only one bridge. By the nature of bridge, it should start somewhere and must return to that point implies a close loop lying on K. Performing an R^*-move, we can reduce a crossing, which is a contradiction that K has minimum number of crossings. Hence, K must be un-knot. This completes the proof ■

In the mathematical field of knot theory, the **tricolorability** of a knot is the ability of a knot to be colored with three different colors subject to certain rules.

***Definition*:**

A knot is tricolorable if each arc of the knot projection can be colored one of three colors, subject to the following rules:

1. At least two colors must be used, and
2. At each crossing, the three incident arcs are either all the same color or all different colors.

Sometimes a projection of a knot can be colored with three colors in such a way that at each crossing either all three colors meet, or there is only one color. If you twist, turn, or pull a projection of a tricolored trefoil, you can still color it with either one color at each crossing, or three. Consequently the trefoil is tricolorable. Many other knots are tricolorable. Instead if you color the projection of figure eight knot, there will be a crossing with exactly two colors. Consequently tricolorability is a way of classifying some of knots.

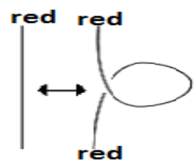
***Theorem 1*:**

Tricolorability is an ambient isotopy invariant

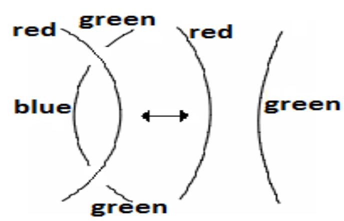
***Proof*:**

We will prove that Reidemeister moves of type I, II and III are tricolorable

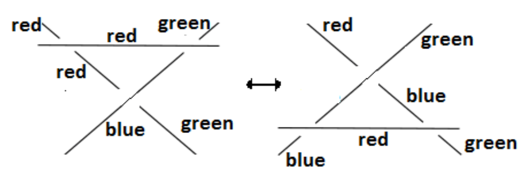
For Reidemeister move of type I, consider;



This verifies that Reidemeister move of type I is tricolorable.
Now,



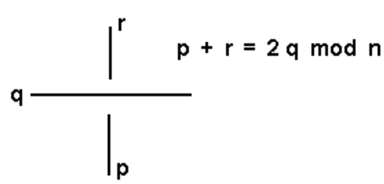
This verifies that Reidemeister move of type II is tricolorable.
Now,



This verifies that Reidemeister move of type III is tricolorable.This completes the proof ■

Definition:

A knot (link) can be colored mod n if integers can be assigned to arcs representing over-crossing (under-crossing) as;



e.g. over-crossing arc = average of under crossing arcss, without the consideration of constant coloring – all strings with same color mod n.

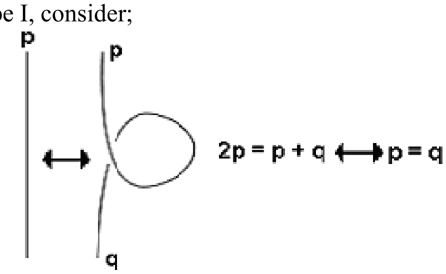
Theorem 2:

For a knot (link) colourability is an ambient isotopy invariant.

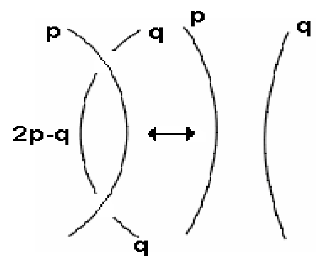
***Proof*:**

We will prove that Reidemeister moves of type I, II and III preserve the coloring.

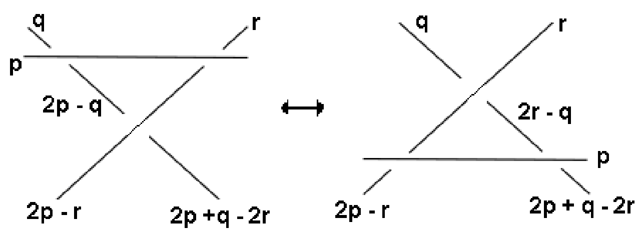
For Reidemeister move of type I, consider;



This verifies that Reidemeister move of type I preserve the coloring.
Now,



It is straight forward to observe that $2p-q=p$ *only if* $p=q \Rightarrow 2p-q$ is not a new color. Hence, the Reidemeister move of type II preserves the coloring.
Now,

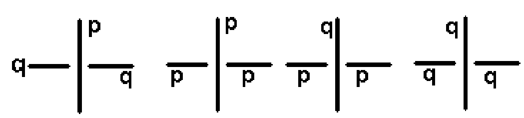


Coloring on R.H.S. will be same if coloring on L.H.S are same and vice versa.

This verifies that Reidemeister move of type III preserve coloring.

This completes the proof■

One can easily observe that the possible coloring at a crossing can only be;



Consequently, a component can have only one color and if there are more than one components the one component can be colored with one color and the other with second color and so on. Hence, we have;

Lemma1:

A knot can not be colored mod 2.

Lemma2:

A link with more than one components is colorable mod 2.

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