Bayesian and Non-Bayesian Estimation of $P(Y < X)$ from Type I Generalized Logistic Distribution Based on Lower Record Values

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Abstract: This paper devoted a Bayesian and non-Bayesian estimation of the stress-strength reliability, $R = P(Y < X)$, when $X$ and $Y$ two independent Type I generalized logistic distribution with common scale parameter. The maximum likelihood estimator and Bayes estimator are proposed for the stress strength reliability based on lower record values. The Bayesian and non-Bayesian confidence intervals for the reliability, $R$, are obtained. Simulation study is conducted to compare and investigate the theoretical results.

Key words: Bayesian and Non Bayesian estimation; Maximum likelihood estimation; Type I generalized logistic distribution; Lower record values; Confidence interval; Stress-Strength.

INTRODUCTION

Balakrishnan and Leung (1988) defined the Type I generalized logistic distribution (Type I GLD) as one of the three generalized forms of the standard logistic distribution. Type I generalized logistic distribution has received additional attention in estimating its parameters for practical usage (see Balakrishnan(1992)). For $a > 0$ and $\lambda > 0$ the two-parameter Type I GLD has the probability density function (pdf) given by

$$f(x, a, \lambda) = a\lambda e^{-\lambda x} (1 + e^{-\lambda x})^{-(a+1)}, \quad -\infty < x < \infty,$$

and has the cumulative distribution function (cdf) given by

$$F(x, a, \lambda) = (1 + e^{-\lambda x})^{-a}, \quad -\infty < x < \infty,$$

where $\lambda$ is the scale parameter and $a$ is the shape parameter. The importance of the logistic distribution is already been left in many areas of human endeavour. Olapade (2000) obtained some properties for the Type I generalized logistic distribution. The two parameter of Type I generalized logistic distribution will be denoted by Type I GL$(a, \lambda)$. The (pdf) in (1.1) has been obtained by compounding an extreme value distribution with a gamma distribution. It is observed by Balakrishnan and Leung (1988) that this distribution is skewed and its kurtosis coefficient is greater than that of the logistic distribution.

In stress-strength model, the stress ($Y$) and the strength ($X$) are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval. Due to the practical point of view of reliability stress-strength model, the estimation problem of $R = P(Y < X)$ has attracted the attention of many authors. This model, first considered by Birnbaum (1956), is commonly used in many engineering applications, such as civil, mechanical and aerospace. Recently Kundu and Gupta (2005) have considered estimation of $R = P(Y < X)$, when $X$ and $Y$ are independent generalized exponential random variables and Raqab and Kundu (2005) considered the case when $X$ and $Y$ are independent generalized Rayleigh random variables. Baklizi (2008) considered the case when $X$ and $Y$ are independent generalized exponential random variables based on lower record values. Several distributions have been used in the literature as failure models. For references see the book by Kotz et al. (2003) or the articles by Church and Harris (1970), Chao (1982) and Kakade et al. (2008).

Record values and associated statistics are of great importance in several real-life applications involving weather, economic and sports data "Olympic records or world records in sport". Also in industry, many products fail under stress, for example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only the record values are observed. Thus, the number of measurements made is considerably smaller than the complete sample size. This "measurement saving" can be important when the measurements of these experiments are costly if the entire sample was destroyed. For more examples, see Gulati and Padgett (1994). There are also situations in which an observation is stored only if it is a record value. These include studies in meteorology, hydrology, seismology, athletic events and mining. In recent years, there has been much work on parametric and nonparametric inference based

The statistical study of record values started with Chandler (1952), he formulated the theory of record values as a model for successive extremes in a sequence of independently and identically random variables. Feller (1966) gave some examples of record values with respect to gambling problems. Resnick (1973) discussed the asymptotic theory of records. Theory of record values and its distributional properties have been extensively studied in the literature for example, see, Ahsanullah (1995), Nagaraja (1988), Arnold and Balakrishnan (1989), and Balakrishnan and Ahsanullah (1994).

The random variables \( L(0) = 1 \) and \( L(m) = \min \{ j: j > L(m - 1); X_j < X_{L(m-1)} \} \) are called the lower record times, and the sequence \( \{L(m), m \geq 0\} \) is called the sequence of lower record times. The sequence \( \{R_m = X_{L(m)}, m \geq 0\} \) is called the sequence of lower record values.

Let \( R_0, R_1, R_2, \ldots, R_m \) represent the first \((m+1)\) lower record values arising from a sequence \( \{X_i\} \) of iid Type I generalized logistic variable. Then the pdf of \( R_m \) is given by (Arnold et al. (1998))

\[
f_{R_m}(r_m) = \frac{1}{\Gamma(m+1)} \left[ -\ln F(r_m) \right]^m f(r_m), \quad r_m > -\infty, \quad m = 0, 1, 2, \ldots, \tag{1.3}
\]

while the likelihood function based on the \((m+1)\) lower record values \( R_0, R_1, R_2, \ldots, R_m \) is given by

\[
f(r_0, r_1, \ldots, r_m) = \prod_{i=0}^{m} \frac{f(r_i)}{\hat{f}(r_i)} f(r_m). \tag{1.4}
\]

where \(-\infty < r_m < r_{m-1} < \ldots < r_1 < r_0 < \infty\).

This paper is devoted to obtain and compare several techniques of estimation based on lower record values for the stress strength reliability, \( R \). Section 2 contains the maximum likelihood estimators for the for the stress strength reliability, \( R \), for the Type I generalized logistic distribution based on lower record values, also, the interval estimation for the reliability, \( R \), is obtained. In Section 3, Bayes estimator for the stress strength reliability, \( R \), is obtained using the squared error loss function and the confidence interval for the reliability, \( R \), is constructed. In Section 4 simulation study for the theoretical results will be provided. Finally some remarks and a brief summary of the results will be concluded in Section 5.

2- Maximum Likelihood Estimator of \( R \):

Let \( X \) be the strength of a system and \( Y \) be the stress acting on it. Then \( X \) and \( Y \) will be random variables from Type I GLD (1.1) with parameters \((\lambda_1, \alpha_1)\) and \((\alpha_2, \alpha_2)\), respectively. Then the reliability function \( R \) is given by

\[
R = P(Y < X) = \int_{-\infty}^{0} \int_{x}^{\infty} f(x)f(y)dydx. \tag{2.1}
\]

If \( \lambda_1 = \lambda_2 = \lambda \), then it is easy to show that the reliability function \( R \) has the following form

\[
R = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \tag{2.2}
\]

Let \( R_0, R_1, R_2, \ldots, R_m \) be the first \((n+1)\) lower record values arising from a sequence \( \{X_i\} \) of iid Type I GLD (1.1) with parameters \((\lambda, \alpha_1)\) and let \( S_0, S_1, S_2, \ldots, S_m \) the first \((m+1)\) lower record values arising from a sequence \( \{Y_j\} \) of iid Type I GLD (1.1) with parameters \((\lambda, \alpha_2)\) where \( \lambda \) is assumed known. Substituting from equations (1.1) and (1.2) in equation (1.4) then the likelihood functions are given by

\[
L_1(\alpha_1, \lambda) \propto \alpha_1^{n+1} e^{-\alpha_1(\lambda r_1)} \omega(\lambda, \gamma) \quad \text{and} \quad L_2(\alpha_2, \lambda) \propto \alpha_2^{m+1} e^{-\alpha_2(\lambda s_1)} \omega(\lambda, \zeta), \tag{2.3}
\]

Where \( \lambda (\lambda r_1) = \ln (1 + e^{-\lambda r_1}) \), \( \kappa(\lambda s_1) = \ln (1 + e^{-\lambda s_1}) \), \( \omega(\lambda, \gamma) = \prod_{i=0}^{m-1} \frac{e^{-\lambda r_i}}{1 + e^{-\lambda r_1}} \) and \( \omega(\lambda, \zeta) = \prod_{i=0}^{m-1} \frac{e^{-\lambda s_i}}{1 + e^{-\lambda s_1}} \).

According to Amin 2012, The maximum likelihood estimators of \( \alpha_1 \) and \( \alpha_2 \) based on lower record values are given by, respectively, \( \hat{\alpha}_1 = \frac{n+1}{\kappa(\lambda r_1)} \) and \( \hat{\alpha}_2 = \frac{m+1}{\kappa(\lambda s_1)} \).

Hence \( \hat{R}_1 \), the maximum likelihood estimator of \( R \), is written as

\[
\hat{R}_1 = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\alpha}_2}. \tag{2.4}
\]
To study the distribution of $\bar{R}_1$, the distribution of $\hat{a}_1$ and $\hat{a}_2$ are needed.

Consider now $\hat{a}_1 = \frac{n+1}{\kappa \lambda r_n}$. Substituting from equations (1.1) and (1.3) the probability density function of $R_n$ is given by

$$f_{R_n}(r_n) = \frac{\lambda \alpha_n^{n+1} e^{-\lambda r_n}}{r_n^{(n+1)}} \left[ \ln (1 + e^{-\lambda r_n}) \right]^n (1 + e^{-\lambda r_n})^{-(\alpha_n+1)}, \quad r_n > -\infty.$$  

Let $Z_1 = \hat{a}_1$ it is easy to show that the probability density function of $Z_1$ is given by

$$f_{Z_1}(z_1) = \frac{(n+1)\alpha_1 z_1^{n+1}}{r_n^{(n+1)\alpha_1}} e^{-\frac{z_1}{r_n}}, \quad z_1 > 0. \quad (2.5)$$

This is recognized as the inverted gamma distribution with parameters $((n+1), (n+1)\alpha_1)$. From the properties of the inverted gamma distribution it is easy to show that

$$\frac{2(n+1)\alpha_1}{z_1} \sim \chi^2_{2(n+1)}. \quad (2.6)$$

Similarly, the probability $Z_2 = \hat{a}_2$ has the inverted gamma distribution with parameters $((m+1), (m+1)\alpha_2)$ and

$$\frac{2(m+1)\alpha_2}{z_2} \sim \chi^2_{2(m+1)}. \quad (2.7)$$

From equations (2.6), (2.7) and by the independence of the two random variables $Z_1$ and $Z_2$ then one can show that

$$\frac{z_2}{z_1} \sim \frac{\alpha_2}{\alpha_1} F_{2(n+1),2(m+1)}. \quad (2.8)$$

Therefore it is easy after some simple transformation that the distribution of $\bar{R}_1$ is that of

$$\frac{2(n+1)\alpha_1}{z_1} \sim \chi^2_{2(n+1)}.$$

where, $F_{2(n+1),2(m+1)}$ is $F$- distribution with $2(n+1), 2(m+1)$ degrees of freedom. This fact can be used to construct the confidence interval for the reliability, $R$. By using some transformations and facts, the $100(1-\tau)$% confidence interval for the stress strength reliability, $R$, based on lower record values is $(L_1, U_1)$ where,

$$L_1 = \left[ 1 + \frac{z_2/z_1}{F_{(1-\tau),2(n+1),2(m+1)}} \right]^{-1} \quad \text{and} \quad U_1 = \left[ 1 + \frac{z_2/z_1}{F_{(1-\tau),2(n+1),2(m+1)}} \right]^{-1}. \quad (2.9)$$

### 3- Bayesian Estimator of $R$:

In recent decades, the Bayesian viewpoint has received frequent attention for analyzing failure data and other time-to-event data, and has been often proposed as a valid alternative to traditional statistical perspectives. Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory. In this section Bayesian estimator $\hat{R}_2$ for $R$ based on lower record values will be discussed. Here, the Bayes estimators are studied under squared error loss function.

Under the assumption that $\lambda_1 = \lambda_2 = \lambda$ and the parameter $\lambda$ is assumed known, the likelihood functions of $\alpha_1$ and $\alpha_2$ based on the two sets of lower record values from the Type I GLD suggest that the conjugate family of prior distributions for $\alpha_1$ and $\alpha_2$ is the gamma distributions (see Baklizi, 2008) where

$$\pi(\alpha_1) \propto \alpha_1^{a_1-1} e^{-b_1\alpha_1}, \quad \alpha_1 > 0 \quad \text{and} \quad \pi(\alpha_2) \propto \alpha_2^{a_2-1} e^{-b_2\alpha_2}, \quad \alpha_2 > 0, \quad (3.1)$$

where $a_1, b_1, a_2, \text{ and } b_2$ are the parameters of prior distributions of $\alpha_1$ and $\alpha_2$ respectively. Combining the likelihood functions (2.3) and the prior density (3.1), then it is easy to show that $a_1 | R \sim \text{Gamma}(n + a_1 + 1, b_1 + \kappa (\lambda r_n))$ and $a_2 | R \sim \text{Gamma}(m + a_2 + 1, b_2 + \kappa (\lambda s_m))$. From the properties of gamma distribution one can show the following;

$$2(b_1 + \kappa (\lambda r_n)) \alpha_1 | R \sim \chi^2_{2(n+a_1+1)}. \quad (3.2)$$
2(b_2 + \kappa(\lambda s_m))\alpha_2 s \sim \chi^2_{2(m+a_2+1)}.

(3.3)

From equations (2.2) and (2.3) the posterior distribution \( \pi(R|\xi, s) \) of the reliability \( R \) is equal to

\[
\left( 1 + \frac{(m+a_2+1)(b_2 + \kappa(\lambda r_n))}{(n+a_1+1)(b_2 + \kappa(\lambda s_m))} \right) F_{2(m+a_2+1),2(n+a_1+1)}^{-1}.
\]

(3.4)

The Bayes estimator is the mean of the posterior distribution (3.4) which may be approximated numerically.

By using the distribution (3.4) and after some simple transformation and facts, the 100(1 - \tau)\% Bayesian confidence interval for the stress strength reliability, \( R \), based on lower record values is \( (L_2, U_2) \) where

\[
L_2 = \left( 1 + AF_{1\frac{1}{2}}(m+a_2+1),2(n+a_1+1) \right)^{-1} \quad \text{and} \quad U_2 = \left( 1 + AF_{1\frac{1}{2}}(m+a_2+1),2(n+a_1+1) \right)^{-1},
\]

(3.5)

where \( A = \frac{(m+a_2+1)(b_2 + \kappa(\lambda r_n))}{(n+a_1+1)(b_2 + \kappa(\lambda s_m))} \).

The case of the non informative prior, consider the Jefferey priors \( \frac{1}{a_1} \) and \( \frac{1}{a_2} \) for \( \alpha_1 \) and \( \alpha_2 \), respectively. After some simple calculation one can show that the posterior distribution of \( R \) is equal to \( \left( 1 + \frac{(m+1)(\kappa(\lambda r_n))}{(n+1)(\kappa(\lambda s_m))} \right) R_{2(m+1),2(n+1)}^{-1} \). Therefore the 100(1 - \tau)\% Bayesian confidence interval for the stress strength reliability, \( R \), based on lower record values is \( (L_3, U_3) \) where

\[
L_3 = \left( 1 + BF_{1\frac{1}{2}}(m+1),2(n+1) \right)^{-1} \quad \text{and} \quad U_3 = \left( 1 + BF_{1\frac{1}{2}}(m+1),2(n+1) \right)^{-1},
\]

(3.6)

where \( B = \frac{(m+1)(\kappa(\lambda r_n))}{(n+1)(\kappa(\lambda s_m))} \).

4 - Simulation Study:

A simulation study is conducted to investigate and compare the performance of various confidence intervals for the stress strength reliability, \( R \) based on lower record values. The software package MathCAD (2001) is used for the simulation study. The following steps are considered:

1- Generate 5000 uniform (0,1) random variables. The usual transformation technique is used to get the corresponding Type I GL random samples. In this way, samples \( x_1, x_2, ..., x_\omega \) from Type I GLD with \( \omega = 100 \).

2- Choose from each vector the first \( (n+1), (n = 3, 5, 7, 10) \) lower record values \( r_0, r_1, r_2, ..., r_n \) for different values of \( \alpha_1 \), when \( \lambda \) is assumed known.

3- By the same manner, generate 5000 uniform (0, 1) random variables. The usual transformation technique is used to get the corresponding Type I GL random samples. In this way, samples \( Y_1, Y_2, ..., Y_\omega \) from Type I GLD with \( \omega = 100 \), and choose from each vector the first \( (m+1), (m = 3, 5, 7, 10) \) lower record values \( s_0, s_1, ..., s_m \) for different values of \( \alpha_2 \) for known value of \( \lambda \).

4- Substituting in equations (2.9), one can obtain the 100(1 - \tau)\% confidence interval for the stress strength reliability \( R \).

5- For given values of prior parameters \( \alpha_1 = \alpha_2 = 2 \), \( b_1 = b_2 = 3 \) and by using equation (3.5) the Bayesian confidence interval for the stress strength reliability with 100(1 - \tau)\% is computed, where \( \tau = 0.05 \).

6- Table 1 contains the simulation results for upper and lower bound for the Bayesian and non-Bayesian confidence intervals for the stress strength reliability \( R \) based on lower record values and for different choices of the distribution parameters \( \alpha_1 \) and \( \alpha_2 \) for different values of sample sizes.

5- Conclusion:

In this paper the estimation problem of the stress-strength reliability \( R = P(Y < X) \) when \( X \) and \( Y \) two independent Type I generalized logistic distribution based on lower record values is discussed. It is assumed that the two populations have the same scale parameters, but different shape parameters. Bayesian and non Bayesian confidence interval for the reliability, \( R \), are obtained. From a simulation study made, the following results are observed:

1- The length of the Bayes confidence interval for the reliability \( R = P(Y < X) \) always shorter than the classical confidence interval.
2. For a fixed value \( n \), the length of confidence interval for the reliability, \( R \), decreases when \( m \) increase.

3. For a fixed value \( m \), the length of confidence interval for the reliability, \( R \), decreases when \( n \) increase.

4. In general, when \( m \) and \( n \) increase the average lengths of all intervals decrease.

### Table 1: Lower and upper bound and the length of Bayesian and non-Bayesian confidence interval for the reliability function.

<table>
<thead>
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<th>Parameters</th>
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<td>( U_1 )</td>
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### REFERENCES


