Some Notes on Ideals in Soft Rings

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Abstract: Molodtsov introduced the concept of soft set theory, which can be used as a generic mathematical tool for dealing with uncertainty. In this paper we will introduce the concepts of soft ideal, prime soft ideals and maximal soft ideals in soft rings and will compare soft maximal ideals and soft prime ideals.

Key words: Prime soft ideal, maximal soft ideal, Idealistic soft prime rings.

Preliminaries:
Researchers in economics, engineering, environmental science, sociology, medical science and many other fields deal daily with the complexities of modeling uncertain data. Classical methods are not always successful, because the uncertainties appearing in these domains may be of various types. While probability theory, fuzzy sets, rough sets, and other mathematical tools are well-known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties as pointed out by Molodtsov. Consequently, Molodtsov proposed a completely new approach for modeling vagueness and uncertainty. This so-called soft set theory is free from the difficulties affecting existing methods. A soft set is a parameterized family of subsets of the universal set. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. This paper begins by introducing the basic concepts of soft ideals in soft rings theory. The main purpose of this paper is to introduce the concepts of soft ideals, prime soft ideals and maximal soft ideals, which extends the notion of a prime ideal in classic ring to include the algebraic structures of soft sets.

INTRODUCTION
In this section, we will recall some basic definitions for soft sets which introduced by Molodtsov. Throughout this paper \( U \) refers to an initial universe set, \( E \) is a set of parameters, \( P(U) \) is the power set of \( U \), \( A \subseteq E \) and \( Z \) is the ring of integer numbers. Molodtsov defined the soft set as the following way.

Definition 1:
A pair \((F,A)\) is called a soft set over \( U \), where \( F \) is a mapping given by \( F: A \to P(U) \). In other words, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For \( \varepsilon \in A \), \( F(\varepsilon) \) may be considered as the set of \( \varepsilon \) – approximate elements of the soft set \((F,A)\). Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [6].

Definition 2:
Assume that \((F,A)\) and \((H,B)\) are two soft sets over a common universe \( U \). We say that \((F,A)\) is a soft subset of \((H,B)\), denoted by \((F,A) \subseteq (H,B)\), if it satisfies the following:
(1) \( A \subseteq B \).
(2) \( F(x) \) and \( H(x) \) are identical approximations for all \( x \in A \).

Definition 3:
Let \((F,A)\) and \((H,B)\) be two soft sets over a common universe \( U \). The intersection of \((F,A)\) and \((H,B)\) is defined as the soft set \((K,C)\) satisfying the following conditions:
(1) \( C = A \cap B \).
(2) For every \( x \in C \), \( K(x) = F(x) \) or \( H(x) \).
In this case, we write \((K,C) = (F,A) \wedge (H,B)\).
Definition 4:
Let \( (F, A) \) and \( (H, B) \) be two soft sets over a common universe \( U \). The bi-intersection of \( (F, A) \) and \( (H, B) \) is defined as the soft set \( (K, C) \) satisfying the following conditions:
1. \( C = A \cap B \).
2. \( K(x) = F(x) \cap H(x) \), for all \( x \in C \).
In this case, we write \( (K, C) = (F, A) \cap (H, B) \).

Definition 5:
Assume that \( (F, A) \) and \( (H, B) \) are two soft sets over a common universe \( U \). The union of \( (F, A) \) and \( (H, B) \) is defined as the soft set \( (K, C) \) satisfying the following conditions:
1. \( C = A \cup B \).
2. For every \( x \in C \), \( K(x) = \begin{cases} F(x) & \text{if } x \in A \setminus B, \\ H(x) & \text{if } x \in B \setminus A, \\ F(x) \cup H(x) & \text{if } x \in A \cap B. \end{cases} \)
In this case, we write \( (K, C) = (F, A) \cup (H, B) \).

Definition 6:
Let \( \{F_{i}, A_{i}\}_{i \in I} \) be a nonempty family of soft sets over a common universe \( U \). The union of these soft sets is defined as the soft set \( (H, C) \) satisfying the following conditions:
1. \( C = \bigcup_{i \in I} A_{i} \).
2. For all \( x \in C \), \( H(x) = \bigcup_{i \in I(x)} F_{i}(x) \) where \( I(x) = \{i \in I \mid x \in A_{i}\} \).
In this case, we write \( \bigcup_{i \in I} (F_{i}, A_{i}) = (H, C) \).

Definition 7:
Assume that \( (F, A) \) is a soft set over a common universe \( U \). The set of all \( x \in A \), where \( F(x) \neq \emptyset \) is called the support of the soft set \( (F, A) \), i.e.
\( \text{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\} \).
Furthermore a soft set \( (F, A) \) is said to be non-null if its support is not equal to the empty set.

Now, we are going to state the definitions of soft ring and soft ideal which is introduced by Acar and et al., in 2010. From now on, \( R \) is a commutative ring and all soft sets are considered over \( R \).

Definition 8:
Let \( (F, A) \) be a non-null soft set over \( R \). Then \( (F, A) \) is said a soft ring over \( R \) if \( F(x) \) is a subring of \( R \), for all \( x \in A \).

Example 1:
Let \( (F, A) \) be a soft set over \( R = Z[x] \), where \( A = Z \) and \( F : A \rightarrow P(Z[x]) \) is a set-valued function defined by \( F(t) = \{f(x) \in Z[x] \mid f_{0} = tq, q \in Z \) and \( f_{0} \) is the constant term of the polynomial \( f \} \)
Then \( F(t) \) is a subring of \( Z[x] \) for all \( t \in A \). Hence, \( (F, A) \) is a soft ring over \( R = Z[x] \).

Definition 9:
Assume that \( (F, A) \) is a soft ring over \( R \). A non-null soft set \( (\emptyset, I) \) over \( R \) is called soft ideal of \( (F, A) \), which will be denoted by \( (\emptyset, I) \trianglelefteq (F, A) \), if it satisfies the following conditions:
1. \( I \subseteq A \).
2. \( \emptyset(x) \) is an ideal of \( F(x) \) for all \( x \) in \( \text{Supp}(\emptyset, I) \).
Example 2:
Assume that \( A = R = Z_6 = \{0, 1, 2, 3, 4, 5\} \) and \( I = \{0, 2, 4\} \). Consider the set-valued function \( F : A \to P(R) \) defined via \( F(x) = \{y \in R | xy \in \{0, 2, 4\}\} \). Therefore, \( F(0) = F(2) = F(4) = Z_6 \) and \( F(1) = F(3) = F(5) = \{0, 2, 4\} \). As we see, all sets are subrings of \( R \) and so \((F, A)\) is a soft ring over \( R \).

Furthermore, consider the function \( \varphi : I \to P(R) \) given by \( \varphi(x) = \{y \in R | x + y \in \{0, 2, 4\}\} \). It is clear that, \( \varphi(0) = \{0, 2, 4\} \triangleleft F(0) = Z_6 \), \( \varphi(2) = \{0, 2, 4\} \triangleleft F(2) = Z_6 \) and \( \varphi(4) = \{0, 2, 4\} \triangleleft F(4) = Z_6 \). Hence, \((\varphi, I)\) is a soft ideal of \((F, A)\).

The Prime Soft Ideal and Maximal Soft Ideal:
In classical algebra, prime ideals and maximal ideals play important role in the structure of a ring. In this section we will introduce the notions of prime soft ideals and soft maximal ideals of a soft ring.

Definition 10:
Assume that \((F, A)\) is a soft ring over \( R \). A soft ideal \((\varphi, I)\) of \((F, A)\) is called a prime soft ideal of \((F, A)\), if \( x \varphi\) is a prime ideal of \( F(x) \) for all \( x \) in \( \text{Supp}(\varphi, I) \).

Example 3:
Let \( A = R = Z_4 = \{0, 1, 2, 3\} \) and \( I = \{0\} \). Suppose that \( F : A \to P(R) \) is the set-valued function defined via \( F(x) = \{y \in R | xy \in \{0, 2\}\} \). Since \( F(0) = F(2) = Z_4 \) and \( F(1) = F(3) = \{0, 2\}, (F, A) \) is a soft ring over \( R \) by Definition 8. Furthermore, by considering the function \( \varphi : I \to P(R) \) given by \( \varphi(x) = \{y \in Z_4 | x + 2y = 0\} \), one has \( \varphi(0) = \{0, 2\} \) is a prime ideal of \( F(0) = Z_4 \) and so \((\varphi, I)\) is a prime soft ideal of soft ring \((F, A)\).

Theorem 1:
Let \((F, A)\) be a soft ring over \( R \) and \((\varphi, I)\), \((\theta, J)\) be prime soft ideals of \((F, A)\) over \( R \). If \( I \) and \( J \) are tow disjoint sets, then \((\varphi, I) \cup (\theta, J)\) is a prime soft ideal of \((F, A)\).

Proof:
By using Definition 5 one can consider \((H, K) = (\varphi, I) \cup (\theta, J)\) where \( K = I \cup J \) and for all \( x \in K \).

\[
H(x) = \begin{cases} 
\varphi(x) & \text{if } x \in I - J, \\
\theta(x) & \text{if } x \in J - I.
\end{cases}
\]

Since \((\varphi, I) \triangleleft (F, A)\) and \((\theta, J) \triangleleft (F, A)\), then \( K \subset A \). Assume that \( x \in \text{Supp}(H, K) \). If \( x \in I - J \), then \( H(x) = \varphi(x) \neq \emptyset \); is a prime ideal of \( F(x) \), since \((\varphi, I)\) is a prime soft ideal of \((F, A)\). Similarly, if \( x \in J - I \), then \( H(x) = \theta(x) \neq \emptyset \); is a prime ideal of \( F(x) \). Hence, \( H(x) \) is a prime ideal of \( F(x) \) for all \( x \in \text{Supp}(H, K) \). Therefore, \((H, K)\) is a prime soft ideal of \((F, A)\) by Definition 10.

Proposition 1:
Let \((F, A)\) be a soft ring over \( R \) and \((\varphi_k, I_k)_{k \in K}\) be a nonempty family of prime soft ideals of \((F, A)\). If \( \{I_k | k \in K\}\) are pair-wise disjoint, then \( \bigcup_{k \in K} (\varphi_k, I_k)\) is a prime soft ideal of \((F, A)\).

Proof:
The proof is as the same of the last theorem.

In classical algebra, for two disjoint prime ideals \( P_1 \) and \( P_2 \) of \( R \), one has \( P_1 \cap P_2 \) is not a prime ideal of \( R \) in general. In the following example we will show that, the same fact holds for soft rings.
Example 4:
Let \( A = R = Z \) and \( I = J = \{2,3\} \). The set-valued function \( F : A \to P(R) \) defined via \( F(x) = \{nx | n \in Z \} = xZ \), gives the soft ring \((F, A)\) over \( R \). Consider the set-valued functions \( \varphi : I \to P(R) \) and \( \theta : J \to P(R) \) given by \( \varphi(x) = \{2nx | n \in Z \} = 2xZ \) and \( \varphi(x) = \{3nx | n \in Z \} = 3xZ \) respectively. One can easily check that, \((\varphi, I)\) and \((\theta, J)\) are prime soft ideals of soft ring \((F, A)\). Assume that \((H, K) = (\varphi, I) \cap (\theta, J)\), where \( K = I \cap J \) and \( H(x) = \varphi(x) \cap \theta(x) \) for all \( x \in K \). Then \( H(2) = \varphi(2) \cap \theta(2) = 4Z \cap 6Z = 12Z \) is not a prime ideal of \( F(2) \) and \( H(3) = \varphi(3) \cap \theta(3) = 6Z \cap 9Z = 18Z \) is not a prime ideal of \( F(3) \). Therefore, \((H, K) = (\varphi, I) \cap (\theta, J)\) is not a prime ideal of \((F, A)\).

Definition 11:
Let \((F, A)\) be a soft ring over \( R \). A soft ideal \((\varphi, I)\) of \((F, A)\) is called a maximal soft ideal of \((F, A)\), if \( \varphi(x) \) is a maximal ideal of \( F(x) \) for all \( x \) in \( \text{Supp}(\varphi, I) \).

Example 5:
In Example 3, \((\varphi, I)\) is a maximal soft ideal of \((F, A)\), since \( \varphi(x) \) is a maximal ideal of \( F(x) \) for all \( x \) in \( \text{Supp}(\varphi, I) \).

Since a prime ideal of a commutative ring \( R \) may be not a maximal ideal, then Every prime soft ideal is not a soft maximal ideal as we show in the following example.

Example 6:
Assume that \( A = R = Z \) and \( I = \{1\} \). By considering the set-valued functions \( F : A \to P(R) \) given by \( F(x) = \{nx | n \in Z \} \) and \( \varphi : I \to P(R) \) given by \( \varphi(x) = \{n(x-1) | n \in Z \} \), one has \( F(x) = xZ \) is a subring of \( R \) for all \( x \in A \) and \( \varphi(1) = \{0\} \) is a prime ideal of \( F(1) = Z \). Therefore \((F, A)\) is a soft ring over \( R \) and \((\varphi, I)\) is its prime soft ideal, but \((\varphi, I)\) is not a maximal soft ideal.

Theorem 2:
Assume that \((F, A)\) is a soft ring over \( R \) and \((\varphi, I)\) is a maximal soft ideal of \((F, A)\). Then there is not any nontrivial soft ideal between \((\varphi, I)\) and \((F, A)\).

Proof:
Let \((\theta, J)\) is a nontrivial soft ideal between \((\varphi, I)\) and \((F, A)\). Since \( I \subseteq J \), there is \( x \in J - I \) such that \( \varphi(x) \subseteq \theta(x) \subseteq F(x) \), which is a contradiction by Definition 11; because \( \varphi(x) \) must be a maximal ideal of \( F(x) \) for all \( x \) in \( \text{Supp}(\varphi, I) \).

Conclusion:
Soft sets are deeply related to fuzzy sets and rough sets. We applied soft sets in ring theory. Hence, by focusing on ideals in soft rings, we have discussed some algebraic properties of soft sets in ring theory and we have introduced the notions of prime soft ideals and maximal soft ideals and have given several examples. To extend this work, one could study on another soft ideals such as soft primary ideals in soft rings.

REFERENCES