Fifth-Order Improved Runge-Kutta Method With Reduced Number of Function Evaluations

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Abstract: The Improved Runge-Kutta (IRK) methods are two step in nature and require lower number of stages per step compared to the classical Runge-Kutta methods. Therefore, the IRK methods will have the lower number of function evaluations per step. Here, the fifth-order Improved Runge-Kutta method (IRK5) with only five stages is derived. The order conditions of the method are obtained up to order six and the coefficients of the fifth order method are determined by minimizing the error norm of the sixth order method. The stability region of the method is presented and numerical examples are given to illustrate the computational efficiency and accuracy of IRK5 compared to RK5.

Key words: Improved Runge-Kutta methods; Two-step methods; Order conditions; Stability region; Ordinary differential equations.

INTRODUCTION

Consider the numerical solution of the initial value problem for the system of ordinary differential equation

\[ y'(x) = f(x, y(x)), \quad x \in [x_0, X], \quad y(x_0) = y_0. \] (1)

One of the most common methods for solving numerically (1) is Runge-Kutta (RK) method. Most efforts to increase the order of RK method have been accomplished by increasing the number of Taylor’s series terms used and thus the number of function evaluations. The RK method of order \( p \) has a local error over the step size \( h \) of \( O(h^{p+1}) \).

Many authors have attempted to increase the efficiency of RK methods with a lower number of function evaluations required. As a result, Goeken and Johnson (2000) proposed a class of Runge-Kutta method with higher derivatives approximations for the third and fourth-order method. Xinyuan (2003) presented a class of Runge-Kutta formulae of order three and four with reduced evaluations of function. Phohomsiri and Udwadia (2004) constructed the Accelerated Runge-Kutta integration schemes for the third-order method using two functions evaluation per step. Udwadia and Farahani (2008) developed the Accelerated Runge-Kutta methods for higher orders. However most of the presented methods are obtained for the autonomous system while the Improved Runge-kutta methods (IRK) can be used for autonomous as well as non-autonomous systems. Rabiei and Ismail (2001) constructed the New Improved Runge-Kutta method with reduced number of function evaluations. The method proposed of order three with two stages. Rabiei and Ismail (2011) developed the third order Improved Runge-Kutta method for solving ordinary differential equations with two and three stages. The third order IRK method with three stages is more accurate compared to the classical third order RK method. Rabiei and Ismail (2011) constructed the Improved Runge-Kutta method for solving ordinary differential equations. The order conditions of the methods up to order five were derived also the convergence and stability region of the methods were discussed. Rabiei and Ismail (2011) developed the fifth-order Improved Runge-Kutta method for solving ordinary differential equations. The method used only five stages. The IRK methods arise from the classical RK methods, can also be considered as a special class of two-step methods. That is, the approximate solution \( y_{n+1} \) is calculated using the values of \( y_n \) and \( y_{n-1} \). Our method introduces the new terms of \( k_{ij} \), which are calculated from \( k_i \), \( i > 2 \) in the previous step. The scheme proposed herein has a lower number of function evaluations than the RK methods while maintaining the same order of local accuracy.

In section 2, we give a general idea of IRK method and order conditions of the method up to order six are given in section 3. In section 4, we derived the fifth order methods with five stages. The stability region of method is presented in section5 and the numerical results and discussions are given in the last section.
General Form of IRK Method:
The general form of the proposed IRK method in this paper with s-stage for solving equation (1) has the form:

\[ y_{n+1} = y_n + h \left( b_1 k_1 - b_1 k_{i-1} + \sum_{j=2}^{s} b_j (k_j - k_{j-1}) \right), \]

for \( 1 \leq n \leq N - 1 \), where

\[ k_i = f(x_n, y_n), \]
\[ k_{i-1} = f(x_{n-1}, y_{n-1}), \]
\[ k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \quad 2 \leq i \leq s, \]
\[ k_{i-1} = f(x_{n-1} + c_i h, y_{n-1} + h \sum_{j=1}^{i-1} a_{ij} k_{j-1}), \quad 2 \leq i \leq s. \]

(2)

for \( c_2, \ldots, c_s \in [0,1] \) and \( f \) depends on both \( x \) and \( y \) while \( k_i \) and \( k_{ij} \) depend on the values of \( k_j \) and \( k_{ij} \) for \( j = 1, \ldots, i - 1 \). Here \( s \) is the number of function evaluations performed at each step and increases with the order of local accuracy of the IRK method. In each step we only need to evaluate the values of \( k_1, k_2, \ldots \), while \( k_{i-1}, k_{i-2}, \ldots \) are calculated from the previous step Figure 1 shows this idea.

![Fig. 1: General construction of IRK method.](image)

The accelerated Runge-Kutta method (see Udwadia, F.E., A. Farahani, 2008), is derived purposely for solving autonomous first order ODEs where the stage or function evaluation involved is of the form

\[ k_i = f(y(x_n + ha, k_{i-1})). \]

where \( k_i \) is a function of \( y \) only and the term involved \( k_{i-1} \). There are two improvement used here, the first one is the function \( f \) is not autonomous, thus the method is not specific for \( y' = f(y(x)) \), but it can be used for solving both the autonomous equations as well as the more general differential equations \( y' = f(x, y(x)) \).

The second improvement is that the internal stages \( k_i \) and \( k_{i-1} \) contain more \( k \) values which are defined as \( \sum_{j=1}^{i-1} a_{ij} k_j \), for \( i = 2, \ldots, s \), compared to the accelerated Runge-Kutta method in which their methods contain
only one $k$ value. This additional $k$ value aimed to make the methods more accurate. Note that IRK method is not self-starting therefore a one-step method must provide the approximate solution of $y^1$ at first step. The one-step method must be of appropriate order to ensure that the difference $y^1 - y(x^1)$ is order of $p$ or higher. For example the Runge-Kutta method is one of the most popular one-step method that we can use to approximate the starting value for IRK5 method. In this paper, without loss the generality we derived the method with $\alpha = 0$, so the IRK method in formulae (2) can be represented as follows:

$$y_{n+1} = y_n + h \left( b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{5} b_j (k_j - k_{-i}) \right),$$

for $1 \leq n \leq N - 1$, where

$$k_1 = f(x_n, y_n),$$
$$k_{-1} = f(x_{n-1}, y_{n-1}),$$
$$k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \quad 2 \leq i \leq s,$$
$$k_{-i} = f(x_{n-1} + c_i h, y_{n-1} + h \sum_{j=1}^{i-1} a_{ij} k_{-j}), \quad 2 \leq i \leq s. \quad (3)$$

For $2 \leq i \leq s$. It is convenient to represent (3) by Table 1.

**Table 1:** Table of coefficients for explicit IRK method ($\alpha = 0$).

<table>
<thead>
<tr>
<th>0</th>
<th>a_{11}</th>
<th>a_{12}</th>
</tr>
</thead>
<tbody>
<tr>
<td>c_2</td>
<td>a_{21}</td>
<td></td>
</tr>
<tr>
<td>c_3</td>
<td>a_{31}</td>
<td>a_{32}</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>c_s</td>
<td>a_{s1}</td>
<td>a_{s2}</td>
</tr>
<tr>
<td>b_0</td>
<td>b_1</td>
<td>b_2</td>
</tr>
</tbody>
</table>

To determine the coefficients of method given by (3), the IRK method expression (3) is expanded using the Taylor’s series expansion. After some algebraic simplifications this expansion is equated to the true solution $y_{n+1}$ at $x_{n+1}$ that is given by Taylor’s series expansion. This result in a system of nonlinear algebraic equations is denoted as order conditions. We try to solve as many order conditions as possible because the highest power of $h$ for which all of the order equations are satisfied is the order of the resulting IRK method. A great deal of algebraic and numeric calculations is required for the above process which was mainly performed using Maple.

**Order Conditions:**

Rabiei and Ismail (2011) derived the order conditions for IRK method up to order five. Here, we developed the Taylor series expansion and obtained the order conditions up to order six as presented in Table 2.

**Fifth-Order Method with Five Stages:**

For $s=5$, the general form of IRK5 are as follows

$$y_{n+1} = y_n + h \left( b_1 k_1 - b_{-1} k_{-1} + \sum_{i=2}^{5} b_j (k_j - k_{-i}) \right),$$
$$k_1 = f(x_n, y_n),$$
$$k_{-1} = f(x_{n-1}, y_{n-1}),$$
$$k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j), \quad 2 \leq i \leq 5,$$
\[ k_{i} = f(x_{n-1} + c_ih, y_{n-1} + h \sum_{j=1}^{i-1} a_{ij} k_{j-1}), \quad 2 \leq i \leq 5. \]  \hspace{1cm} (4)

For \( 2 \leq i \leq 5 \), \( c_i \in [0, 1] \) and \( c_i = \sum_{j=1}^{i-1} a_{ij} \). Also we can represent the coefficients of IRK5 method by Table 3.

**Table 2:** Order conditions of IRK method up to order six.

<table>
<thead>
<tr>
<th>Order of Method</th>
<th>Order Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>First order</td>
<td>( b_1 - b_{-1} = 1 ).</td>
</tr>
<tr>
<td>Second order</td>
<td>( b_{-1} + \sum_{i} b_i = \frac{1}{2} ).</td>
</tr>
<tr>
<td>Third order</td>
<td>( \sum_{i} b_i c_i = \frac{5}{12} ).</td>
</tr>
<tr>
<td>Fourth order</td>
<td>( \sum_{i} b_i c_i^2 = \frac{1}{3} ), ( \sum_{ij} b_i a_{ij} c_j = \frac{1}{6} ).</td>
</tr>
<tr>
<td>Fifth order</td>
<td>( \sum_{i} b_i c_i^3 = \frac{31}{120} ), ( \sum_{ij} b_i a_{ij} c_j c_k = \frac{31}{240} ), ( \sum_{ij} b_i a_{ij} c_j^2 = \frac{31}{360} ), ( \sum_{ijk} b_i a_{ij} a_{jk} c_k = \frac{31}{720} ).</td>
</tr>
<tr>
<td>Sixth order</td>
<td>( \sum_{i} b_i c_i^4 = \frac{1}{5} ), ( \sum_{ij} b_i c_i^2 a_{ij} c_j = \frac{1}{10} ), ( \sum_{ijk} b_i a_{ij} c_j c_k = \frac{1}{20} ), ( \sum_{ijk} b_i a_{ij} c_j^2 = \frac{1}{15} ), ( \sum_{ijk} b_i a_{ij} c_j = \frac{1}{20} ), ( \sum_{ijkl} b_i a_{ij} a_{jk} c_k = \frac{1}{30} ), ( \sum_{ijkl} b_i a_{ij} a_{jk} c_k^2 = \frac{1}{40} ), ( \sum_{ijkl} b_i a_{ij} a_{jk} c_k^2 = \frac{1}{40} ), ( \sum_{ijkl} b_i a_{ij} a_{jk} a_{km} c_m = \frac{1}{120} ).</td>
</tr>
</tbody>
</table>

**Table 3:** Table of coefficients for IRK5 method.

<table>
<thead>
<tr>
<th>0</th>
<th>c_2</th>
<th>c_3</th>
<th>c_4</th>
<th>c_5</th>
<th>b_1</th>
<th>b_2</th>
<th>b_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_{21}</td>
<td>a_{31}</td>
<td>a_{32}</td>
<td>a_{41}</td>
<td>a_{51}</td>
<td>b_1</td>
<td>b_2</td>
<td>b_3</td>
</tr>
</tbody>
</table>

To determine the coefficients of IRK5, they need to satisfy the order conditions of the method up to order five (see Table 2). Here, we choose \( c_2 = \frac{1}{4} \), \( c_3 = \frac{1}{4} \), \( c_4 = \frac{1}{2} \), \( c_5 = \frac{3}{4} \) and obtained the values of \( b_{-1}, b_1, b_2, b_3, b_4, b_5 \) depend on other parameters by solving the first sixth equations of order conditions from Table 2. We have
\[ b_{-1} = \frac{1}{45}, \quad b_1 = \frac{46}{45}, \]

\[ b_2 = -\frac{1}{90}a_{32} (6a_{32} - 58a_{52} + 60 + 9a_{42} + 9a_{43} - 58a_{53} - 116a_{54}), \]

\[ b_3 = -\frac{1}{90}a_{32} (-58a_{52} + 60 + 9a_{42} + 9a_{43} - 58a_{53} - 116a_{54}), \]

\[ b_4 = \frac{1}{10}, \quad b_5 = \frac{29}{45}. \]

Substitute the values of \( b_{-1}, b_1, b_2, b_3, b_4, b_5 \) in the following reminder order conditions for the fifth order method.

\[ \sum_{i=3}^{5} \sum_{j=2}^{i-1} b_i c_i a_{ij} c_j = \frac{31}{240}, \]

\[ \sum_{i=3}^{5} \sum_{j=2}^{i-1} b_i a_{ij} c_j^2 = \frac{31}{360}, \]

\[ \sum_{i=4}^{5} \sum_{j=3}^{i-1} \sum_{k=2}^{j-1} b_i a_{ij} a_{jk} c_k = \frac{31}{720}. \]

By choosing \( a_{32}, a_{43} \) and \( a_{53} \) as free parameters we have

\[ a_{42} = \frac{9}{32} a_{43}a_{32} - \frac{29}{16} a_{53}a_{32} - a_{43} + \frac{31}{64}, \]

\[ a_{54} = \frac{151}{7424} + \frac{81}{3712} a_{43}a_{32} - \frac{9}{64} a_{53}a_{32} - a_{53}, \]

\[ a_{54} = \frac{16}{29}. \]

To minimize the error norm of the sixth order method we substitute all the parameters into as many as possible order conditions for the sixth order method. Here, we have to sets of coefficients of IRK5 method which are mentioned as set1 and set2. We choose the second and third equations from the order conditions of the sixth order method for set1 and for set2 we choose the sixth and eighth equations.

For set1 we have:

\[ f_2 = \sum_{i=3}^{5} \sum_{j=2}^{i-1} b_i c_i^2 a_{ij} c_j - \frac{1}{10}, \]

\[ f_3 = \sum_{i=4}^{5} \sum_{j=3}^{i-1} \sum_{k=2}^{j-1} b_i a_{ij} c_j a_{jk} c_k - \frac{1}{20}, \]

Define

\[ \tau = \sqrt{f_2^2 + f_3^2}. \]

By minimizing the value of \( \tau \) we obtained the values of free parameters using Maple software.
\[ \tau = 9.59 \times 10^{-18}, \]
\[ a_{32} = 0.7322, a_{43} = 3.34482, a_{53} = 0.0559. \]

Substituting the free parameters we obtained the values of other coefficients which are presented as set1 in table 4.

**Table 4: Set1 coefficients of IRK5 method.**

<table>
<thead>
<tr>
<th>0</th>
<th>0.25</th>
<th>-0.2272</th>
<th>0.7322</th>
<th>-2.2485</th>
<th>3.344</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.5734</td>
<td>0.0121</td>
<td>0.0559</td>
<td>0.5517</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1750</td>
<td>0.0222</td>
<td>0.0961</td>
<td>0.0295</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

For set2 we have:

\[ f_6 = \sum_{i=4}^{5} \sum_{j=3}^{i-1} b_i c_i a_{ij} a_{jk} c_k - \frac{1}{30}, \]
\[ f_6 = \sum_{i=4}^{5} \sum_{j=3}^{i-1} b_i a_{ij} a_{jk} c_k^2 - \frac{1}{60}. \]

Define

\[ \tau = \sqrt{f_6^2 + f_6^2}. \]

By minimizing the value of \( \tau \) we obtained the values of free parameters as follows:

\[ \tau = 2.57 \times 10^{-20}, \]
\[ a_{32} = 0.2586, a_{43} = 0.6444, a_{53} = 0.8918. \]

Substituting the free parameters we obtained the values of other coefficients which are presented as set2 in table 5.

**Table 5: Set2 coefficients of IRK5 method.**

<table>
<thead>
<tr>
<th>0</th>
<th>0.25</th>
<th>-0.0082</th>
<th>0.2586</th>
<th>-0.5312</th>
<th>0.6444</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.25</td>
<td>0.3860</td>
<td>-0.9002</td>
<td>0.8918</td>
<td>0.5517</td>
</tr>
<tr>
<td>0.75</td>
<td>0.2060</td>
<td>0.0403</td>
<td>-0.1070</td>
<td>-0.1</td>
<td>0.6444</td>
</tr>
<tr>
<td>0.0222</td>
<td>1.0222</td>
<td>0.0403</td>
<td>-0.1070</td>
<td>-0.1</td>
<td>0.6444</td>
</tr>
</tbody>
</table>

In section 6, to illustrate the efficiency of our methods we compared the numerical results of new methods with the classical Butcher’s Runge-Kutta method of order five which are denoted as RK5 method in Table 6 (see Butcher, J.C., 1964).

**Stability Region:**

Generally, to define the stability region of the method given in (3) we applied the test problem \( y' = \lambda y \), where \( \lambda \) is a complex number (See (Rabiei, F., F. Ismail, 2001; Rabiei, F., F. Ismail, 2011; Rabiei, F., F. Ismail, 2011). Here, for \( \lambda = 5 \) after applying the test problem to (4) we have the stability polynomial as follows:

\[ \rho(\xi, \tilde{h}) = \xi^2 - \left( \frac{1}{120} \tilde{h}^5 + \frac{31}{720} \tilde{h}^4 + \frac{1}{6} \tilde{h}^3 + \frac{5}{12} \tilde{h}^2 + \frac{2}{3} \tilde{h} + 1 \right) \xi + \frac{1}{120} \tilde{h}^5 + \frac{31}{720} \tilde{h}^4 + \frac{1}{6} \tilde{h}^3 + \frac{5}{12} \tilde{h}^2 + \frac{2}{3} \tilde{h}. \] (5)

Stability region of the methods is the set of values of \( \tilde{h} \) such that all the roots of stability polynomial are inside the unit circle. Here, the stability region of IRK5 method is plotted in Figures 2.
### Table 6: Coefficients of Butcher’s RK5.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1/4</th>
<th>1/2</th>
<th>4/5</th>
<th>1</th>
<th>1/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>1/4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/2</td>
<td>1/5</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4/5</td>
<td>0</td>
<td>-1/2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3/32</td>
<td>0</td>
<td>0</td>
<td>9/32</td>
<td>1</td>
<td>-7/32</td>
<td>8</td>
</tr>
<tr>
<td>2/3</td>
<td>16/3</td>
<td>2</td>
<td>12</td>
<td>-7/32</td>
<td>7/32</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>90</td>
<td>0</td>
<td>32</td>
<td>12</td>
<td>32</td>
<td>7</td>
<td>90</td>
</tr>
</tbody>
</table>

**Fig. 2:** Stability region of IRK5 for $\lambda = h$.

In addition the stability regions of the proposed method are close to the stability region of the fifth-order Accelerated Runge-Kutta method (see (Udwadia, F.E., A. Farahani, 2008).

**Numerical Examples:**

In this section, we tested a standard set of initial value problems to show the efficiency and accuracy of the proposed methods. The exact solution $y(x)$ is used to estimate the global error as well as to approximate the starting value of $y_1$ at the first step $[x_0, x_1]$. The following problems are solved for $x \in [0, 10]$.

**Problem 1 (an oscillatory problem, see Hull, T.E., et al., 1980)**

\[ y' = y \cos x, \quad y(0) = 1, \]

Exact solution: $y(x) = e^{\sin x}$.

**Problem 2 (see [11])**

\[ y_1' = -2y_1 + y_2 + 2 \sin x, \quad y_1(0) = 2, \]
\[ y_2' = y_1 - 2y_2 + 2(\cos x - \sin x), \quad y_2(0) = 3, \]

Exact solution: $y_1(x) = 2e^{-x} + \sin x$, $y_2(x) = 2e^{-x} + \cos x$.

The number of function evaluations versus the -log (maximum global error) for the tested problems are shown in Figure 3-6.
**Fig. 3:** Number of function evaluations versus of maximum global error for problem 1 (set1).

**Fig. 4:** Number of function evaluations versus of maximum global error for problem 1 (set2).

**Fig. 5:** Number of function evaluations versus of maximum global error for problem 2 (set1).
Discussion and Conclusion:

From Figures 3 - 6 we observed that for both tested problems the new method has slightly lower number of function evaluations compared to the existing RK5 method.

As a conclusion, the fifth order improved Runge-Kutta methods have been developed for numerical integration of first order ordinary differential equations with reduced number of function evaluations required per step. The order conditions of the new method are derived up to order six and by satisfying the appropriate order conditions we obtained the two sets of coefficients for the method of order five which used only five-stages. The stability region of the method is given. The IRK5 method is almost two-step in nature and it is as competitive as the existing classical RK5 method.

REFERENCES


