Fuzzy Stochastic Linear Programming Problems With Uncertainty Probability Distribution

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Abstract: We focused in this study on fuzzy stochastic linear programming problems, in which the coefficients of the objective, constraint and right hand side parameters are triangular fuzzy random variables under uncertainty probability distribution. Two-phase solution strategy of the problem have been developed are the de-fuzzification of the problem via ranking function to its stochastic counterpart and the conversion of the stochastic problem to deterministic problem via stochastic transformation. The results illustrated by an example.

Key words: Fuzzy stochastic linear programming, ranking function, optimization problem, uncertainty.

INTRODUCTION

Recently solving fuzzy stochastic optimization problems, has been attracted more attention by researchers, and the general strategy while facing these types of the optimization problems is to defuzzify and derandomize fuzzy random variables and convert the original problem to its deterministic form and solve it(Luhandjula, 2006; Hop, 2007a).

In many real states, one think that there are true probabilities distributions describe the dissimilar random phenomenal exist, but not able to determine their strict values, so must deal with those phenomena as a fuzziness and randomness simultaneously. Since linear programming is one of the maximum significant tool, but one of the most important applicable tool operational research in the real life world optimization problems (Lilien, 1987; Meyer zuSelhausen, 1989; Tingley, 1987), hence dealing with such problems as a fuzzy stochastic linear programming problems under uncertainty are more appropriate to get a solution to such optimization model?

The fuzzy concept, fuzzy objective functions and constraints and what related it introduced for the first time by Bellman and Zadeh (1970), then others studied in the fuzzy environment.Zimmermann (1978) was presented early works of fuzzy programming (Zimmermann, 1983) and Zhong et al. (1994) studied fuzzy random linear programming.Cadenas and Verdegay (2000) considered a multiobjective linear programming problems, when coefficients of the objective functions are given as a fuzzy numbers, and used ranking functions in it. An interactive method has been provided for solving multiobjective fuzzy stochastic programming problems (Mohan and Nguyen, 2001). Iskander (2003) utilized weighted objective function to convert a multiobjectivestochastic fuzzy linear programming problem. On the other hand he studied another type of fuzzy stochastic fractional linear programming problems under uncertainty (Iskander, 2004b), furthermore a fuzzy weighted additive approach used to stochastic fuzzy goal programming and possibility and necessity suggested in stochastic fuzzy linear programming too (Iskander, 2004a; Iskander, 2005).

Stochastic programming studied by Ben Abdelaziz and Masri (2005) considered that a probability distribution can be defined as crisp or fuzzy inequalities, used α-cut technique to defuzzify the probability distribution.Nasseri et al. (2005) does not converted a linear programming problems to their crisp equivalent while used ranking function to solve them.

Many credit references which co-occur both fuzziness and randomness in a linear programming problems related fuzzy stochastic did survey in invited review by Luhandjula (2006). For fuzzy stochastic goal programming problem where coefficients of objective, constraints and goals are fuzzy random variables, Hop (2007a) presented a model to measure attainment value to solve this kind of problems(Hop, 2007c; Hop, 2007b).

Rommelfanger (2007) considered linear multicriteria programming problems with crisp, fuzzy or stochastic Proposed concepts to solve it, emphasized that some data modeled by fuzzy sets, both sides of the constraints should have the same fuzzy numbers with or fuzzy intervals, and proposed a procedure for solve it. Sakawa et al. (2012b) considered fuzzy random two level linear programming problems with vagueness judgments of decision makers, introduced fuzzy goals into formulated non-cooperative problem including fuzzy random variables and has considered Stackelberg solutions for decision making problems in hierarchical optimizations in fuzzy random environments. On the other hand they (Sakawa et al., 2012a)introduced several concepts like: hierarchical modeling and structures, fuzziness and randomness simultaneously, no cooperative relationships between two decision makers are taken into account in dealing with bi-level linear programming problems,
considering non-cooperation between decision makers, and assuming that both possibility and necessity measure is fulfilled by each objective function. Ullah Khan et al. (2012) studied fully fuzzy linear programming problems, proposed a simplified novel technique for solving it employed a modified version well-known simple method.

In these circumstances, we study fuzzy stochastic linear programming problems, the problem is solved in two stages first a ranking function is used to defuzzify the problem to convert it into stochastic ones, then transformed it into deterministic problem and solving it.

Fuzzy stochastic Linear Programming Problems under fuzzy linear partial information on probability distribution

The mathematical formulation of a Fuzzy stochastic Linear Programming Problems with incomplement on probability distribution (FSLPPPI) can be written as follows:

\[
\begin{align*}
\text{Min } C^{-1}(w).x &= \text{Min}\{C_1^{-1}(w).x, \ldots , C_k^{-1}(w).x\}, \\
\text{s.t.} & \quad T^{-}(w).x - h^{-}(w) \geq 0, \quad \forall x \in X,
\end{align*}
\]

where \( X_0 = \{x \in R^n : A.x = b_x, x \geq 0\} \) is the set of deterministic constraints with \( A_x \) a \( m_x \times n \) matrix and \( b_x \) a \( m_x \) vector, \( C^{-}\), \( T^{-} \) and \( h^{-} \) are triangular fuzzy random matrices of \( k \times n \), \( m \times n \), and \( m \times 1 \) respectively defined on some probability space \( (\Omega, \mathcal{F}, P) \) with \( \Omega = \{w_1, \ldots , w_N\} \) is a discrete set of events, \( 2^\Omega \) is the power set of \( \Omega \) and \( P \) is the partially known probability distribution that signs to each \( A \in 2^\Omega \) the probability of occurrence \( P(A) \) (Ben Abdelaziz and Masri, 2005; Ben Abdelaziz and Masri, 2010). Each probability \( p_i \) of a given even \( w_i, i = 1, \ldots , N \) belongs to the known polyhedral set:

\[
\pi = \left\{ p = (p_1, p_2, \ldots , p_N) : Ap \leq b, \sum_{i=1}^{N} p_i = 1, p_i \geq 0, i = 1,2, \ldots , N \right\}
\]

where \( A = a_{ij} \) and \( b = b_i \) are respectively \( s \times N \) and \( s \times 1 \) and fixed matrices.

To solve formulation (1), we have to use the fuzzy transformation first and then the stochastic transformation, to get stochastic programming and then convert to deterministic linear programming problems, and then finding a solution to the problem, as we detail in the next sections.

**Fuzzy transformation:**

The aim of fuzzy transformation is to defuzzify the fuzzy stochastic coefficients in the objective function, so as the matrix constraint coefficients in addition to its right hand parameter. There are some membership function and \( \alpha \)-cut techniques to get this aim as used by many researchers. But also there is another technique to reach that goal, named ranking function. Now we are going to insert definition of the fuzzy number and its ranking function.

**Definition (1):**

Let \( A^{-} = (a^L, a^U, a, \beta) \) denote the trapezoidal fuzzy number, where \( [a^L - a, a^U + \beta] \) is the support of \( A \) and \( [a^L, a^U] \) its modal set (Mahdavi-Amiri and Nasseri, 2006).

The set of all trapezoidal fuzzy numbers can be denoted by \( F(R) \). If \( a = a^L = a^U \), then a triangular fuzzy number will be obtained, and can show it with \( A = (a, a, \beta) \).

Mahdavi-Amiri and Nasseri (2006), Mahdavi-Amiri and Nasseri (2007), and Yager (1981) introduced a linear ranking function that is similar to the ranking function adopted by Maleki (2002), and they defined it as follows:

**Definition (2):** For a trapezoidal fuzzy number \( R(a^{-}) = (a^L, a^U, a, \beta) \) the ranking function can use as follows:

\[
R(a^{-}) = \int_{a}^{a^U}(\inf a^- + \sup a^-)da^L.
\]

This reduces to:

\[
R(a^{-}) = \frac{a^L + a^U}{2} + \frac{a^- - a^L}{4}.
\]

Since we have triangular fuzzy numbers in the matrices as considered in (1), it is a special case in trapezoidal fuzzy numbers as emphasized in the definition (1), hence a triangular fuzzy number is represented by a triple \( T^- = (u_1, u_2, u_3) \) and is defined as (Kaufmann and Gupta, 1988; Bector and Chandra, 2005):
There are basic arithmetic operations on the triangular numbers. On the other hand by combining definitions (1, 2) together the triple represented (Kaufmann and Gupta, 1988; Bector and Chandra, 2005), a ranking function on a triangular fuzzy sets as a mapping from a fuzzy set to the set of real numbers with four basic properties (Ullah Khan et al., 2012; Nasseri et al., 2005) can be defined as follows:

\[
\mu_T(x) = \begin{cases} 
0 & \text{if } x < u_1 \\
\frac{x - u_3}{u_2 - u_1} & \text{if } u_1 \leq x \leq u_2 \\
\frac{u_3 - x}{u_3 - u_2} & \text{if } u_2 \leq x \leq u_3 \\
0 & \text{if } x > u_3
\end{cases}
\] (3)

There are basic arithmetic operations on the triangular numbers. On the other hand by combining definitions (1, 2) together the triple represented (Kaufmann and Gupta, 1988; Bector and Chandra, 2005), a ranking function on a triangular fuzzy sets as a mapping from a fuzzy set to the set of real numbers with four basic properties (Ullah Khan et al., 2012; Nasseri et al., 2005) can be defined as follows:

\[
R(T^+) = u_2 + \frac{u_3 - 2u_2 + u_1}{4}
\] (4)

The ranking triangular fuzzy numbers has been used to defuzzify fuzziness system, thus we are going to employ (4) to defuzzify (1) based on Nasseri et al. (2005) we obtain the following system:

\[
\text{Min } C^i(w). x = \text{Min}[C^1_i(w). x, ..., C^k_i(w). x],
\]

\[
\text{s.t.: } T(w). x - h(w) \geq 0,
\]

\[
x \in X,
\]

where \( C, T \) and \( h \) are random matrices, and the other details remain as described in (1).

**Certainty of a probability:**

In many situations in our real life world we faced in-complement or uncertainty probability distribution. So we receive expert ambiguous information on a probability distribution, thus when get the assertion of the expert situation of \( p_i, 1 \leq i \leq N \), like \( p_i \) is around \( \alpha \in (0,1) \) that means we have the collection of the set of the fuzzy probability distribution on the form of:

\[
\pi^- = \left\{ p = (p_1, p_2, ..., p_N) : Ap \leq b, \sum_{i=1}^{N} p_i = 1, p_i \geq 0, i = 1, 2, ..., N \right\}
\] (6)

where \( A, \) and \( b \) are as defined in (2). Thus we have the stochastic linear programming with fuzzy partial information on probability distribution. In other words the formulation (6) is (2) where we know the probability distribution. To convert (6) into (2), we have to defuzzify it either by using opportunemembership function if we know (supposing)the vagueness level and the decision maker credibility or if the probability distribution in the form of (3), can use (4) to get the certainty of the probability.

**Stochastic transformation:**

As emphasized by Stancu-Minasian (1984), Ben Abdelaziz et al. (1999) and Ben Abdelaziz (2012) the single (multiobjective) stochastic programming is not mathematically well defined. So they pointed out two transformations to resolution such types of problems: multiobjective transformation and stochastic transformation. In the first one the problem is transformed as a multiobjective deterministic optimization problem and then solved it as an interactive method, while in the second one multiobjective collected to obtain a unique stochastic problem and then stochastic programming approach employed to solve the problem.

**Random Constraints And Their Transformations:**

To solve (1), by using whatever method or transforming, first the random constraints should be addressed and transformed, for achieve that aim the chance constrained programming approach can be used with a certain probability (Charnes and Cooper, 1963). Thus:

\[
P(T(w). x - h(w) \geq 0) \geq \beta
\] (7)

where \( \beta \) acceptable threshold value by the decision maker. The lower probability function related to \( p \) has been considered by Ben Abdelaziz and Masri (2005) and Ben Abdelaziz and Masri (2010), as a chance constraint of the problem, because for impossibility for satisfying for unlimited number of \( P \). Now, can combine the expected value of the objective function of (5) with (7) by employed (2) to get the deterministic problem as follows:

\[
\text{Min } \text{Max } E_{p \in \pi} C^i. x = \text{Min } \text{Max } E_{p \in \pi}[c^1_i. x, ..., c^k_i. x]
\]

\[
\text{s.t.: } \quad P(T(w). x - h(w) \geq 0) \geq \beta, \forall p \in \pi
\]

\[
x \in X,
\]

The model of (8) is deterministic and can solve LINDO or LINGO.
Recourse Approach:

two stage transforming(fuzzy and stochastic) have been done and If a practicable assessment for deterministic restraints does not verifies the indeterminate restrictions (deviancy deviation or lack shortage happens) to the problem in the formula (8) then the obtained solution should be penalizes by presenting a recourse function as an additional cost purpose in the objective function.

Solution algorithm

Step1: Consider the problem as the mathematical form in (9).
Step2: Use (4) to defuzzify the coefficients in the objective function, both sides of the constraints and uncertain probability distribution.
Step3: Repeat Step1, Step2 for the recourse approach (10).
Step4: Use the stochastic transformation to get the deterministic optimization problem as in (11). Step5: Use LINGO software to get optimum solution of (11).

Illustrative example:

Consider the following production example for producing two products $P_1$ and $P_2$ by mixing different raw materials $M_1$ and $M_2$:

$$\begin{align*}
\text{Min } c_1^-(w).x_1 + c_2^-(w).x_2 \\
\text{s.t.:}
\end{align*}$$

$$a_{11}^-(w).x_1 + a_{12}^-(w).x_2 \geq b_1^-(w)$$

$$a_{21}^-(w).x_1 + a_{22}^-(w).x_2 \geq b_2^-(w)$$

$$x_1 + x_2 \leq 18$$

$$x_i \geq 0, i = 1,2$$

(9)

where

1. $x_1$: Quantity purchased of the $M_1$; $x_2$: quantity purchased of the $M_2$.
2. $w$: The random variable that describes the possible situation of the market, takes three states: $w_1$, $w_2$ and $w_3$.
3. $c_i^-(w), i = 1, 2$: The fuzzy triangular price of purchasing and treatment of the of$x_i, i = 1, 2$ and the coefficients depend on the market states.
4. $b_j^-(w), j = 1, 2$: The fuzzy triangular quantity of command of the products $P_1$ and $P_2$. The demands depend on the states of the market.
5. $a_{ij}^-(w)$: The fuzzy triangular quantity of the $M_i$ needs for the production of the $P_j$. This parameter also depends on the states of the market.
6. The capacity of the treatment expressed by $x_1 + x_2 \leq 18$. So, the decision making can expressed as follows; $X = \{(x_1, x_2) \in R^2, s.t. x_1 + x_2 \leq 18; x_i \geq 0, i = 1, 2\}$.

The different value of the parameter of the problem (9) given in the following table (1):

<table>
<thead>
<tr>
<th>Parameter P</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1^-(w)$</td>
<td>(50,52,53)</td>
<td>(58,60,61)</td>
<td>(68,70,75)</td>
</tr>
<tr>
<td>$c_2^-(w)$</td>
<td>(60,61,63)</td>
<td>(61,65,70)</td>
<td>(68,70,73)</td>
</tr>
<tr>
<td>$a_{11}^-(w)$</td>
<td>(2,2,1,2,5)</td>
<td>(2,2,2,2,4)</td>
<td>(1,6,1,8,2)</td>
</tr>
<tr>
<td>$a_{12}^-(w)$</td>
<td>(0,9,1,1,3)</td>
<td>(0,9,1,1,1)</td>
<td>(1,2,1,5,1,6)</td>
</tr>
<tr>
<td>$a_{21}^-(w)$</td>
<td>(1,1,1,2,1,3)</td>
<td>(1,1,3,1,5)</td>
<td>(0,7,0,9,1)</td>
</tr>
<tr>
<td>$a_{22}^-(w)$</td>
<td>(1,1,2,1,8)</td>
<td>(2,2,2,2,3)</td>
<td>(2,1,2,3,2,4)</td>
</tr>
<tr>
<td>$b_1^-(w)$</td>
<td>(30,33,45)</td>
<td>(28,31,32)</td>
<td>(20,21,22)</td>
</tr>
<tr>
<td>$b_2^-(w)$</td>
<td>(15,18,20)</td>
<td>(20,23,25)</td>
<td>(20,21,22)</td>
</tr>
</tbody>
</table>

Ifa demand of products $P_1$ or $P_2$ is not satisfied the penalty cost is added to the model (9), the penalty cost $q_1^-(w)$ and$q_2^-(w)$ for the products $P_1$ and $P_2$ respectively, depends also on the states of the market, as showed in the following table:

<table>
<thead>
<tr>
<th>Parameter P</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1^-(w)$</td>
<td>(55,58,60)</td>
<td>(63,65,70)</td>
<td>(70,72,73)</td>
</tr>
<tr>
<td>$r_2^-(w)$</td>
<td>(58,60,62)</td>
<td>(68,70,72)</td>
<td>(70,73,75)</td>
</tr>
</tbody>
</table>
When the first judgment is made, to Purchas the amount of the $M_1$ and $M_2$ and when the situations of the market is happens, the recourse function can be assumed to fulfilled to demands $P_1$ and $P_2$, so $(y_1, y_2)$ may be given as follows:

$$R(x_1, x_2, w_i) = \min r_1^{-1}(w_i)y_1 + r_2^{-1}(w_i)y_2$$

s.t.: $y_1 - y_3 = b_1^{-1}(w_i) - [a_{11}^{-1}(w_i)x_1 + a_{12}^{-1}(w_i)x_2]$ (10)

$$y_2 - y_4 = b_2^{-1}(w_i) - [a_{21}^{-1}(w_i)x_1 + a_{22}^{-1}(w_i)x_2]$$

$y_i; i = 1, ..., A$

On the other hand the probability distribution to the market is described as a fuzzy states numbers as follows:

$p_1 = (0.24, 0.35, 0.44), p_2 = (0.41, 0.5, 0.6), p_3 = (0.1, 0.2, 0.27)$.

Solution: we defuzzify by using (4), get the following data;

Table 3: The parameter and its states

<table>
<thead>
<tr>
<th>Parameter $P$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1^{-1}(w)$</td>
<td>51 %</td>
<td>59 %</td>
<td>70 %</td>
</tr>
<tr>
<td>$c_2^{-1}(w)$</td>
<td>61 1/3</td>
<td>65 1/3</td>
<td>70 1/3</td>
</tr>
<tr>
<td>$a_{11}^{-1}(w)$</td>
<td>2 7/40</td>
<td>2 1/5</td>
<td>1 4/5</td>
</tr>
<tr>
<td>$a_{12}^{-1}(w)$</td>
<td>1 1/10</td>
<td>1</td>
<td>1 9/20</td>
</tr>
<tr>
<td>$a_{21}^{-1}(w)$</td>
<td>1 1/5</td>
<td>1 11/40</td>
<td>7/8</td>
</tr>
<tr>
<td>$a_{22}^{-1}(w)$</td>
<td>1 3/10</td>
<td>2 7/40</td>
<td>2 11/40</td>
</tr>
<tr>
<td>$b_1^{-1}(w)$</td>
<td>32 ½</td>
<td>30 ½</td>
<td>32</td>
</tr>
<tr>
<td>$b_2^{-1}(w)$</td>
<td>17 ½</td>
<td>22 ½</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 4: The penalty cost parameter and its states

<table>
<thead>
<tr>
<th>Parameter $P$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1^{-1}(w)$</td>
<td>57 ¾</td>
<td>65 ¾</td>
<td>71 ¾</td>
</tr>
<tr>
<td>$r_2^{-1}(w)$</td>
<td>60</td>
<td>70</td>
<td>72 ½</td>
</tr>
</tbody>
</table>

On the other side the probability distribution is $(p_1, p_2, p_3) = \frac{69}{200}, \frac{291}{400}, \frac{77}{400}$. Now by finding the expectation value for each parameter in both of the table (3) and (4), and employ (10) at (9) we get the deterministic problem as follows:

$$\min \frac{15321}{320}x_1 + \frac{11437}{212}x_2 + \frac{11493}{217}y_1 + \frac{447}{8}y_2$$

s.t.: $\frac{837}{451}x_1 + \frac{641}{500}x_2 \geq \frac{213}{6}$

$\frac{135}{128}x_1 + \frac{279}{181}x_2 \geq \frac{28099}{1600}$

$\frac{135}{128}x_1 + \frac{279}{181}x_2 + y_1 - y_3 = \frac{213}{8}$

$\frac{135}{128}x_1 + \frac{279}{181}x_2 + y_2 - y_4 = \frac{28099}{1600}$

$x_1 + x_2 \leq 18$

$x_i \geq 0, y_i \geq 0; i = 1, 2, i = 1, ..., A$.

(11)

Using LINDO we get the result:

Table 5: LP Optimum solution

<table>
<thead>
<tr>
<th>Objective function</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$y_4$</th>
<th>Iteration</th>
<th>Elapsed time</th>
</tr>
</thead>
<tbody>
<tr>
<td>759.5897</td>
<td>13.2385</td>
<td>2.33103</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>00:00:02</td>
</tr>
</tbody>
</table>

Conclusion:

The fuzzy stochastic linear programming problems has been considered in which objective coefficients, constraint and right hand side parameters are triangular fuzzy random variables under uncertainty probability distribution. The problem has been defuzzify via ranking function to its stochastic counterpart and the conversion of the stochastic problem to deterministic problem via stochastic transformation. The results showed that the proposal technique is practical, applicable, takes less iterations and elapsed time.
REFERENCES


