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Solution and Behavior of a Rational Recursive Sequence of order Four

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Abstract: We obtain in this paper the solutions of the following recursive sequences

$x_{n+1}=x_{n-2}\ x_{n-3}/x_n(\pm1\pm x_{n-2}\ x_{n-3})$, n=0,1,...

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solutions.

Key words: difference equations, recursive sequences, stability, periodic solution.

INTRODUCTION

Difference equations or discrete dynamical systems is diverse field which impact almost every branch of pure and applied mathematics . Every dynamical system $a_{n+1}=f\,(a_n)$ determines a difference equation and vise versa . Recently ,there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology , economic, probability theory, genetics, psychology, ...etc. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points.

In this paper we obtain the solutions of the following recursive sequences

$$x_{n+1}=x_{n-2}\ x_{n-3}/x_n(\pm1\pm x_{n-2}\ x_{n-3}) \ , \quad n=0,1,... \tag{1}$$

where the initial conditions are arbitrary real numbers. Also, we study the behavior of the solutions.

Here, we recall some notations and results which will be useful in our investigation.

Let *I* be some interval of real numbers and let $f:I^{k+1}\rightarrow I,$ be a continuously differentiable function. Then for every set of initial conditions

$x_{-k},x_{-k+1},...,x_0\in I,$ the difference equation

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),\ n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ (Kocic, V.L. and G. Ladas, 1993).

Definition 1. (Equilibrium Point)

A point $\overline{x}\in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x}=f(\overline{x},\overline{x},...,\overline{x}).$$

That is, $x_n=\overline{x}$ for $n\geq0$,is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f .

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Definition 2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ with

$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$

we have

$|x_n - x| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$

we have

$\lim_{n\to\infty} x_n = \overline{x}.$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$

we have

$\lim_{n\to\infty} x_n = \overline{x}.$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f\ (\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}$$

Theorem A [33]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0 \quad , n = 0, 1, ...$$

Definition 3. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

The nature of many biological systems naturally leads to their study by means of a discrete variable.

Particular examples include population dynamics and genetics. Some elementary models of biological phenomena, including a single species population model, harvesting of fish, the production of red blood cells, ventilation volume and blood CO_2 levels, a simple epidemics model and a model of waves of disease that can be analyzed by difference equations are shown in (Mickens, R.E., 1987). Recently, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass (1977). In that paper they investigated a simple first order difference-delay equation that models the concentration of blood-level CO_2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow.

The dynamical characteristics of population system have been modelled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations.

In practice, one can formulate a discrete model directly from experiments and observations. Sometimes, for numerical purposes one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points (Kulenovic, M.R.S. and G. Ladas, 2001). But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is desirable that a differential equation, when derived from a difference equation, preserves the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics and bifurcation behaviors. If such discrete models can be derived from continuous-time models and it will preserve the considered realities; such discrete-time models can be called `dynamically consistent' with the continuous-time models.

The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formulation and analysis of discrete time systems, the numerical integration of differential equations by finite-difference schemes, the study of deterministic chaos, etc. For example, (Ladas, G., 1989) the study of oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1}=N_n \exp[r(1-\sum_{j=0}^{m} p_j N_{n-j}],$$

where $r,\ p_m \in (0,\infty), p_0, p_1, ..., p_{m-1} \in [0,\infty)$ and $m+r \neq 1,$ which describes situations where population growth is not continuous but seasonal with non-overlapping generations, leads to the study of oscillations about zero of a linear difference equation of the form

$$x_{n+1}-x_n+\sum_{i=0}^{m} p_i x_{n-k_i}=0, \quad n=0,1,...$$

Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations.

For example, the difference equation,

$$y_{n+1}=y_n \exp[r(1-y_n/k)],$$

has been used to model various animal populations. This equation is considered by some to be the discrete analogue of the logistic differential equation

$y'(t)=ry(t)(1-y(t)/k)$,

where r and k are the growth rate and the carrying capacity of population, respectively.

Grove, *et al*. (2000) studied the stability and the semicycles of solutions of the biological model:

$$x_{n+1}=ax_n+bx_{n-1}e^{-x_n}$$

El-Metwally *et al*. (2003) investigated the asymptotic behavior of the population model:

$$x_{n+1}=\alpha+\beta x_{n-1}e^{-x_n}$$

where α is the immigration rate and β is the population growth rate.

Ding *et al*. (2008) studied the following discrete delay Mosquito population equation

$$x_{n+1}=(\alpha x_n+\beta x_{n-1})e^{-x_n}$$

The generalized Beverton-Holt stock recruitment model has investigated in (DeVault, R., 1998; Beverton, R.J. and S.J. Holt, 1957):

$$x_{n+1}=ax_n+(bx_{n-1})/(1+cx_{n-1}+dx_n)$$

See also (Chen, J. and D. Blackmore, 2002; Elabbasy, E.M., 2007; Elettreby, M.F. and H. El-Metwally, 2007; El-Metwally, H., 2007; El-Metwally, H. and M.M. El-Afifi, 2008; El-Metwally, H., 2001; Grove, E.A., 2000; Huo, H.F. and W.T. Li, 2005; Pielou, E.C., 1974; Kuang, Y. and J.M. Cushing, 1996; Rafiq, A., 2006; Saleh, M. and S. Abu-Baha, 2006; Sedaghat, H., 2003; Summers, D., 2000). The long term behavior of the solutions of nonlinear difference equations of order greater than one has been extensively studied during the last decade. For example, various results about boundedness, stability and periodic character of the solutions of the second-order nonlinear difference equation see (Abu-Saris, R., 2008; Agarwal, R.P., 2000; Agarwal, R.P. and E.M. Elsayed, 2008; Aloqeili, M., 2006; Aloqeili, M., 2006; Atalay, M., 2005; Beverton, R.J. and S.J. Holt, 1957; Chen, J. and D. Blackmore, 2002; Cinar, C., 2004; DeVault, R., 1998; Ding, X. and R. Zhang, 2008; Elabbasy, E.M., 2007; Elabbasy, E.M. and E.M. Elsayed, 2008; Elabbasy, E.M., 2007; Elabbasy, E.M., 2006; Elabbasy, E.M., 2007; Elabbasy, E.M., 2007).

Many researchers have investigated the behavior of the solution of difference equations for example: Agarwal *et al*. (Agarwal, R.P. and E.M. Elsayed, 2008) investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1}=a+(dx_{n-l}x_{n-k})/(b-cx_{n-s})$$

Aloqeili (2006) has obtained the solutions of the difference equation

$$x_{n+1}=x_{n-1}/(a-x_nx_{n-1})$$

Cinar (2004) investigated the solutions of the following difference equation

$$x_{n+1}=ax_{n-1}/(1+bx_nx_{n-1})$$

Elabbasy *et al*. (Elabbasy, E.M., 2006; Elabbasy, E.M., 2007) investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equations

$$x_{n+1}=ax_n-bx_n/(cx_n-dx_{n-1}),$$

$$x_{n+1}=\alpha x_{n-k}/(\beta+\gamma\sum_{i=1}^{k}x_{n-i})$$

Ibrahim (Ibrahim, T.F., 2009) get the solutions of the rational difference equation

$$x_{n+1}=x_n x_{n-2}/x_{n-1}(a+bx_n x_{n-2})$$

Karatas *et al*. (2006) get the form of the solution of the difference equation
x_{n+1}=((x_{n-5})/(1+x_{n-2}x_{n-5}))
Other related results on rational difference equations can be found in refs. (Elsayed, E.M., 2009; Grove, E.A., 2000; Grove, E.A. and G. Ladas, 2005; Grove, E.A., 2000; Hamza, A.E., S.G. Barbary, 2008; Huo, H.F. and W.T. Li, 2005; Ibrahim, T.F., 2009; Karatas, R., 2006; Kocic, V.L. and G. Ladas, 1993; Yalçınkaya, I. and C. Cinar, 2009; Zayed, E.M.E. and M.A. El-Moneam, 2005).

On the Recursive Sequence $x_{n+1}=x_{n-2}\ x_{n-3}/x_n(1+x_{n-2}\ x_{n-3})$

In this section we give a specific form of the solution of the equation in the form

$$x_{n+1}=x_{n-2}\ x_{n-3}/x_n(1+x_{n-2}\ x_{n-3})\quad ,\qquad n=0,1,... \tag{3}$$

where the initial values are arbitrary positive real numbers.

Theorem Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(3). Then for $n=0,1,...$

$$x_{6n-3}=d\prod_{i=0}^{n-1}\frac{(1+2iab)(1+(2i+1)bc)(1+2icd)}{(1+(2i+1)ab)(1+(2i)bc)(1+(2i+1)cd)}$$

$$x_{6n-2}=c\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)(1+(2i)bc)(1+(2i+1)cd)}{(1+2iab)(1+(2i+1)bc)(1+(2i+2)cd)}$$

$$x_{6n-1}=b\prod_{i=0}^{n-1}\frac{(1+(2i)ab)(1+(2i+1)bc)(1+(2i+2)cd)}{(1+(2i+1)ab)(1+(2i+2)bc)(1+(2i+1)cd)}$$

$$x_{6n}=a\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)(1+(2i+2)bc)(1+(2i+1)cd)}{(1+(2i+2)ab)(1+(2i+3)bc)(1+(2i+2)cd)}$$

$$x_{6n+1}=\frac{cd}{a(1+cd)}\prod_{i=0}^{n-1}\frac{(1+(2i+2)ab)(1+(2i+1)bc)(1+(2i+2)cd)}{(1+(2i+1)ab)(1+(2i+2)bc)(1+(2i+3)cd)}$$

$$x_{6n+2}=\frac{ab(1+cd)}{d(1+bc)}\prod_{i=0}^{n-1}\frac{(1+(2i+1)ab)(1+(2i+2)bc)(1+(2i+3)cd)}{(1+(2i+2)ab)(1+(2i+3)bc)(1+(2i+2)cd)}$$

where $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_0=a$.

Proof:
For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$x_{6n-9}=d\prod_{i=0}^{n-2}\frac{(1+2iab)(1+(2i+1)bc)(1+2icd)}{(1+(2i+1)ab)(1+(2i)bc)(1+(2i+1)cd)}$$

$$x_{6n-8}=c\prod_{i=0}^{n-2}\frac{(1+(2i+1)ab)(1+(2i)bc)(1+(2i+1)cd)}{(1+2iab)(1+(2i+1)bc)(1+(2i+2)cd)}$$

$$x_{6n-7} = b\prod_{i=0}^{n-2}\frac{\left(1+\left(2i\right)ab\right)\left(1+\left(2i+1\right)bc\right)\left(1+\left(2i+2\right)cd\right)}{\left(1+\left(2i+1\right)ab\right)\left(1+\left(2i+2\right)bc\right)\left(1+\left(2i+1\right)cd\right)}$$

$$x_{6n-6} = a\prod_{i=0}^{n-2}\frac{\left(1+\left(2i+1\right)ab\right)\left(1+\left(2i+2\right)bc\right)\left(1+\left(2i+1\right)cd\right)}{\left(1+\left(2i+2\right)ab\right)\left(1+\left(2i+3\right)bc\right)\left(1+\left(2i+2\right)cd\right)}$$

$$x_{6n-5} = \frac{\mathrm{cd}}{\mathrm{a(1+cd)}}\prod_{i=0}^{n-2}\frac{\left(1+\left(2i+2\right)ab\right)\left(1+\left(2i+1\right)bc\right)\left(1+\left(2i+2\right)cd\right)}{\left(1+\left(2i+1\right)ab\right)\left(1+\left(2i+2\right)bc\right)\left(1+\left(2i+3\right)cd\right)}$$

$$x_{6n-4} = \frac{\mathrm{ab(1+cd)}}{\mathrm{d(1+bc)}}\prod_{i=0}^{n-2}\frac{\left(1+\left(2i+1\right)ab\right)\left(1+\left(2i+2\right)bc\right)\left(1+\left(2i+3\right)cd\right)}{\left(1+\left(2i+2\right)ab\right)\left(1+\left(2i+3\right)bc\right)\left(1+\left(2i+2\right)cd\right)}$$

Now, it follows from Eq.(3) and the above assumptions that

$$x_{6n-3}=x_{6n-6}x_{6n-7}/x_{6n-4}(1+x_{6n-6}x_{6n-7})$$

$$=d\prod_{i=0}^{n-1}\frac{\left(1+2iab\right)\left(1+\left(2i+1\right)bc\right)\left(1+2icd\right)}{\left(1+\left(2i+1\right)ab\right)\left(1+\left(2i\right)bc\right)\left(1+\left(2i+1\right)cd\right)}$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem

Eq.(3) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.

Proof:

For the equilibrium points of Eq.(3), we can write

$$\bar{x}=\bar{x}^2/\bar{x}(1+\bar{x}^2)$$

Then we have

$$\bar{x}^2(1+\bar{x}^2)=\bar{x}^2$$

$$\bar{x}^2(1+\bar{x}^2-1)=0,$$

or,

$$\bar{x}^4=0.$$

Thus the equilibrium point of Eq.(3) is $\bar{x}=0$.

Let $f:(0,\infty)^3\to(0,\infty)$ be a function defined by

$f(u,v,w)=((vw)/(u(1+vw)))$

Therefore it follows that

$f_{u}(u,v,w)=-((vw)/(u^2(1+vw)))$, $f_{v}(u,v,w)=(w/(u(1+vw)^2))$, $f_{w}(u,v,w)=(v/(u(1+vw)^2))$,

we see that

$f_{u}(x,x,x)=-1$, $f_{v}(x,x,x)=1$, $f_{w}(x,x,x)=1$.

The proof follows by using Theorem A.

Numerical Examples:

For confirming the results of this section, we consider numerical examples which represent different types of solutions to Eq. (3).

Example 1. We assume $x_{-3}=6$, $x_{-2}=2$, $x_{-1}=10$, $x_0=7$ See Fig. 1.

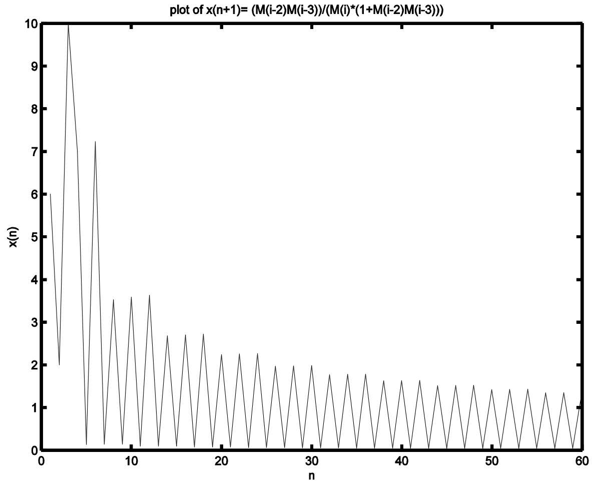

Fig. 1:

Example 2. See Fig. 2, since $x_{-3}=0.8$, $x_{-2}=0.6$, $x_{-1}=0.5$, $x_0=1.3$.

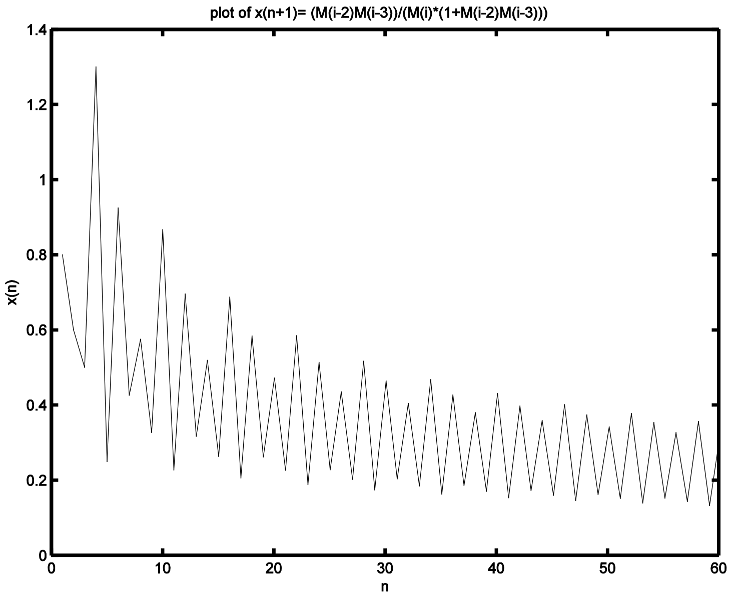

Fig. 2:

On the Recursive Sequence $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1+x_{n-2}x_{n-3})}$

In this section we obtain the solution of the difference equation in the form

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1+x_{n-2}x_{n-3})},\quad n=0,1,..., \qquad (4)$$

where the initial values are arbitrary non zero real numbers with $x_{-3}x_{-2}\neq 1$, $x_{-2}x_{-1}\neq 1$, $x_{-1}x_0\neq 1$.

Theorem *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(4). Then the solution of Eq.(4) is given by the following formula for $n = 0,1,2,\ldots$*

$$x_{6n-3} = \frac{d(-1+bc)^n}{(-1+ab)^n(-1+cd)^n}, \quad x_{6n-2} = \frac{c(-1+ab)^n(-1+cd)^n}{(-1+bc)^n},$$

$$x_{6n-1} = \frac{b(-1+bc)^n}{(-1+ab)^n(-1+cd)^n}, \quad x_{6n} = \frac{a(-1+ab)^n(-1+cd)^n}{(-1+bc)^n},$$

$$x_{6n+1} = \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}}, \quad x_{6n+2} = \frac{ab(-1+ab)^n(-1+cd)^{n+1}}{d(-1+bc)^{n+1}},$$

where $x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$x_{6n-9} = \frac{d(-1+bc)^{n-1}}{(-1+ab)^{n-1}(-1+cd)^{n-1}}, \quad x_{6n-8} = \frac{c(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}},$$

$$x_{6n-7} = \frac{b(-1+bc)^{n-1}}{(-1+ab)^{n-1}(-1+cd)^{n-1}}, \quad x_{6n-6} = \frac{a(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}},$$

$$x_{6n-5} = \frac{cd(-1+bc)^{n-1}}{a(-1+ab)^{n-1}(-1+cd)^n}, \quad x_{6n-4} = \frac{ab(-1+ab)^{n-1}(-1+cd)^n}{d(-1+bc)^n}.$$

Now, it follows from Eq.(4) that

$$x_{6n-3} = \frac{x_{6n-6}x_{6n-7}}{x_{6n-4}(-1+x_{6n-6}x_{6n-7})}$$

$$= \frac{\frac{a(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}}\frac{b(-1+bc)^{n-1}}{(-1+ab)^{n-1}(-1+cd)^{n-1}}}{\left(\frac{ab(-1+ab)^{n-1}(-1+cd)^n}{d(-1+bc)^n}\right)\left(-1+\frac{a(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}}\frac{b(-1+bc)^{n-1}}{(-1+ab)^{n-1}(-1+cd)^{n-1}}\right)}$$

$$= \frac{ab}{\left(\frac{ab(-1+ab)^{n-1}(-1+cd)^n}{d(-1+bc)^n}\right)(-1+ab)} = \frac{d(-1+bc)^n}{(-1+ab)^{n-1}(-1+cd)^n(-1+ab)}.$$

Hence, we have

$$x_{6n-3} = \frac{d(-1+bc)^n}{(-1+ab)^n(-1+cd)^n}.$$

Similarly

$$x_{6n-2} = \frac{x_{6n-5}x_{6n-6}}{x_{6n-3}(1+x_{6n-5}x_{6n-6})}$$

$$= \frac{\frac{cd(-1+bc)^{n-1}}{a(-1+ab)^{n-1}(-1+cd)^n}\frac{a(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}}}{\left(\frac{d(-1+bc)^n}{(-1+ab)^n(-1+cd)^n}\right)\left(1+\frac{cd(-1+bc)^{n-1}}{a(-1+ab)^{n-1}(-1+cd)^n}\frac{a(-1+ab)^{n-1}(-1+cd)^{n-1}}{(-1+bc)^{n-1}}\right)}$$

$$= \frac{\frac{cd}{(-1+cd)}}{\left(\frac{d(-1+bc)^n}{(-1+ab)^n(-1+cd)^n}\right)\left(1+\frac{cd}{(-1+cd)}\right)} = \frac{c(-1+ab)^n(-1+cd)^n}{(-1+bc)^n(-1+cd)\left(1+\frac{cd}{(-1+cd)}\right)}.$$

Hence, we have

$$x_{6n-2} = \frac{c(-1+ab)^n(-1+cd)^n}{(-1+bc)^n}.$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem *Eq.(4) has three equilibrium points which are* $0, \pm\sqrt{2}$ *.and these equilibrium points are not locally asymptotically stable.*

Proof: For the equilibrium points of Eq.(4), we can write

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}(-1+\bar{x}^2)}.$$

Then we have

$$\bar{x}^2(-1+\bar{x}^2) = \bar{x}^2,$$

or

$$\bar{x}^2(\bar{x}^2-2) = 0,$$

Thus the equilibrium points of Eq.(4) are $0, \pm\sqrt{2}$.

Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{vw}{u(-1+vw)}.$$

$$f_u(u,v,w) = -\frac{vw}{u^2(-1+vw)},\quad f_v(u,v,w) = \frac{-w}{u(-1+vw)^2},\quad f_w(u,v,w) = \frac{-v}{u(-1+vw)^2},$$

we see that

$$f_u(\bar{x},\bar{x},\bar{x}) = \pm 1,\quad f_v(\bar{x},\bar{x},\bar{x}) = -1,\quad f_w(\bar{x},\bar{x},\bar{x}) = -1.$$

The proof follows by using Theorem A.

Lemma 1. It is easy to see that every solution of Eq.(4) is unbounded except in the following case.

Theorem *Eq.(4) has a periodic solution of period six iff* $ab = bc = cd = 2$ *and will be taken the form* $\left\{d, c, b, a, \frac{cd}{a}, \frac{ab}{d}, d, c, b, a, \frac{cd}{a}, \frac{ab}{d}, \ldots\right\}$.

Proof: First suppose that there exists a prime period six solution

$$d, c, b, a, \frac{cd}{a}, \frac{ab}{d}, d, c, b, a, \frac{cd}{a}, \frac{ab}{d}, \ldots,$$

of Eq.(4), we see from the form of the solution of Eq.(4) that

$$d = \frac{d(-1+bc)^n}{(-1+ab)^n(-1+cd)^n},\quad c = \frac{c(-1+ab)^n(-1+cd)^n}{(-1+bc)^n},$$

$$b = \frac{b(-1+bc)^n}{(-1+ab)^n(-1+cd)^n},\quad a = \frac{a(-1+ab)^n(-1+cd)^n}{(-1+bc)^n},$$

$$\frac{cd}{a} = \frac{cd(-1+bc)^n}{a(-1+ab)^n(-1+cd)^{n+1}},\quad \frac{ab}{d} = \frac{ab(-1+ab)^n(-1+cd)^{n+1}}{d(-1+bc)^{n+1}}.$$

Then we get

$$-1+ab = -1+cd = -1+bc = 1.$$

Thus

$$ab = bc = cd = 2.$$

Second assume that $ab = bc = cd = 2$. Then we see from the form of the solution of Eq.(4) that

$$x_{6n-3} = d,\quad x_{6n-2} = c,\quad x_{6n-1} = b,\quad x_{6n} = a,\quad x_{6n+1} = \frac{cd}{a},\quad x_{6n+2} = \frac{ab}{d},$$

Thus we have a periodic solution of period six and the proof is complete.

Numerical examples

Here we will represent different types of solutions of Eq. (4).

Example 3. We consider $x_{-3} = 1.8,\ x_{-2} = 1.6,\ x_{-1} = 2.5,\ x_0 = 1.3.$ See Fig. 3.

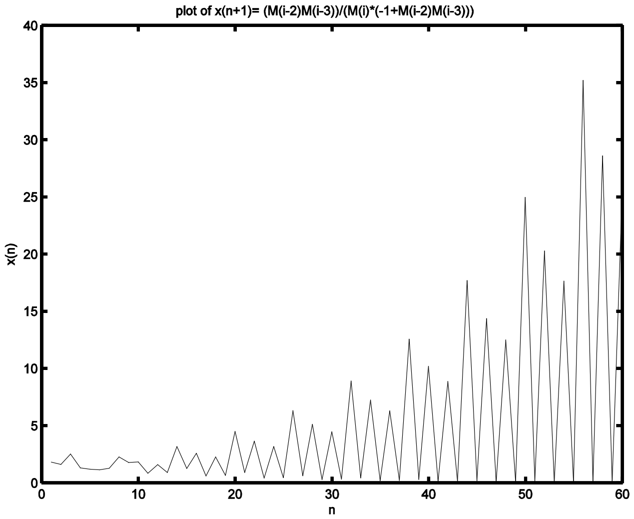

Fig. 3:

Example 4. See Fig. 4, since we suppose that $x_{-3}=11$, $x_{-2}=3$, $x_{-1}=2/3$, $x_0=3$.

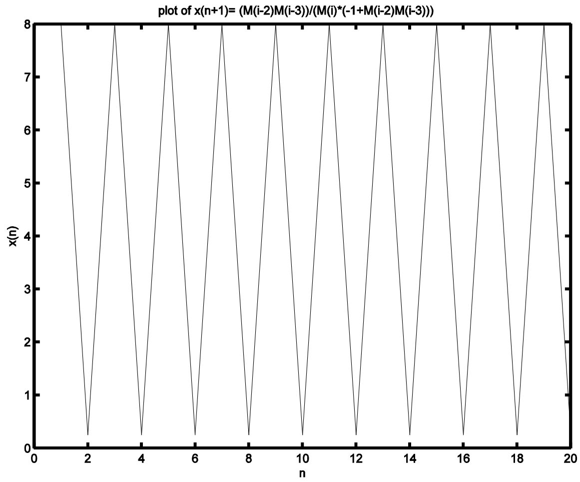

Fig. 4:

The following cases can be proved similarly.

On the Recursive Sequence $x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_{n}(1-x_{n-2}x_{n-3})}$

In this section we get the solution of the third following equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_{n}(1-x_{n-2}x_{n-3})},\quad n=0,1,..., \tag{5}$$

where the initial values are arbitrary positive real numbers.

Theorem *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of Eq.(5). Then for* $n = 0, 1, \ldots$

$$x_{6n-3} = d\prod_{i=0}^{n-1}\left(\frac{(1-(2i)ab)(1-(2i+1)bc)(1-(2i)cd)}{(1-(2i+1)ab)(1-(2i)bc)(1-(2i+1)cd)}\right),$$

$$x_{6n-2} = c\prod_{i=0}^{n-1}\left(\frac{(1-(2i+1)ab)(1-(2i)bc)(1-(2i+1)cd)}{(1-(2i)ab)(1-(2i+1)bc)(1-(2i+2)cd)}\right),$$

$$x_{6n-1} = b\prod_{i=0}^{n-1}\left(\frac{(1-(2i)ab)(1-(2i+1)bc)(1-(2i+2)cd)}{(1-(2i+1)ab)(1-(2i+2)bc)(1-(2i+1)cd)}\right),$$

$$x_{6n} = a\prod_{i=0}^{n-1}\left(\frac{(1-(2i+1)ab)(1-(2i+2)bc)(1-(2i+1)cd)}{(1-(2i+2)ab)(1-(2i+1)bc)(1-(2i+2)cd)}\right),$$

$$x_{6n+1} = \frac{cd}{a(1-cd)}\prod_{i=0}^{n-1}\left(\frac{(1-(2i+2)ab)(1-(2i+1)bc)(1-(2i+2)cd)}{(1-(2i+1)ab)(1-(2i+2)bc)(1-(2i+3)cd)}\right),$$

$$x_{6n+2} = \frac{ab(1-cd)}{d(1-bc)}\prod_{i=0}^{n-1}\left(\frac{(1-(2i+1)ab)(1-(2i+2)bc)(1-(2i+3)cd)}{(1-(2i+2)ab)(1-(2i+3)bc)(1-(2i+2)cd)}\right),$$

where $x_{-3} = d,\ x_{-2} = c,\ x_{-1} = b,\ x_0 = a.$

Theorem *Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Example 5. Assume that $x_{-3} = 11,\ x_{-2} = 8,\ x_{-1} = 4,\ x_0 = 10$ see Fig. 5.

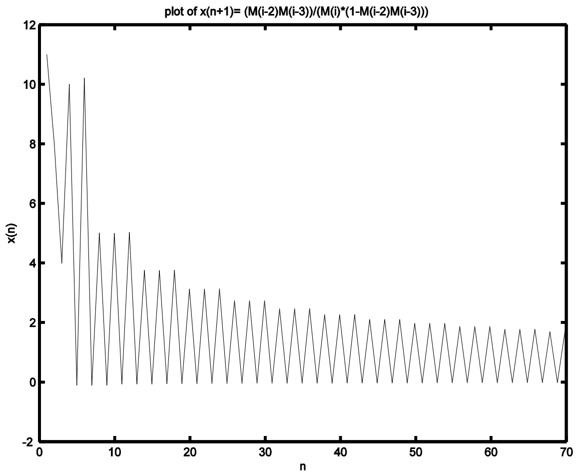

Fig. 5:

Example 6. Assume that $x_{-3}=0.7$, $x_{-2}=1.2$, $x_{-1}=0.4$, $x_0=0.5$ see Fig. 6

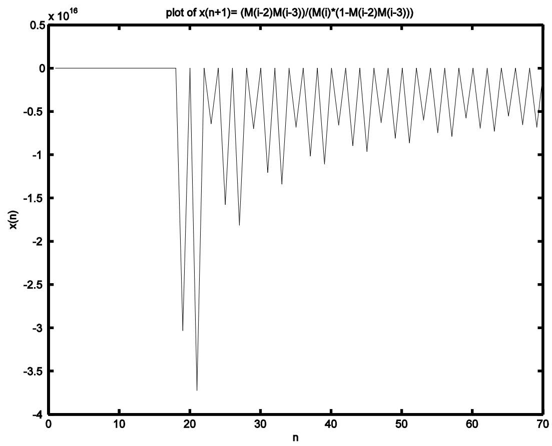

Fig. 6:

On the Recursive Sequence x_{n+1}=((x_{n-2}x_{n-3})/(x_{n}(-1-x_{n-2}x_{n-3})))
Here we obtain a form of the solutions of the equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}}{x_n(-1-x_{n-2}x_{n-3})},\quad n=0,1,\ldots, \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_{-3}x_{-2}\neq -1$, $x_{-2}x_{-1}\neq -1$, $x_{-1}x_0\neq -1$.

Theorem *Let $\{x_n\}_{n=-3}^{\infty}$ be a solution of Eq.(6). Then for $n=0,1,2,\ldots$ the solution of Eq.(6) is given by*

$$x_{6n-3}=\frac{d(-1-bc)^n}{(-1-ab)^n(-1-cd)^n},\quad x_{6n-2}=\frac{c(-1-ab)^n(-1-cd)^n}{(-1-bc)^n},$$

$$x_{6n-1}=\frac{b(-1-bc)^n}{(-1-ab)^n(-1-cd)^n},\quad x_{6n}=\frac{a(-1-ab)^n(-1-cd)^n}{(-1-bc)^n},$$

$$x_{6n+1}=\frac{cd(-1-bc)^n}{a(-1-ab)^n(-1-cd)^{n+1}},\quad x_{6n+2}=\frac{ab(-1-ab)^n(-1-cd)^{n+1}}{d(-1-bc)^{n+1}},$$

where $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_0=a$.

Theorem *Eq.(6) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Lemma 2. It is easy to see that every solution of Eq.(6) is unbounded except in the following case.

Theorem *Eq.(6) has a periodic solution of period six iff $ab=bc=cd=-2$ and will be taken the form $\left\{d,c,b,a,\frac{cd}{a},\frac{ab}{d},d,c,b,a,\frac{cd}{a},\frac{ab}{d},\ldots\right\}$.*

Example 7. Fig. 7. shows the solutions when $x_{-3}=1.7$, $x_{-2}=-0.2$, $x_{-1}=0.4$, $x_0=0.5$.

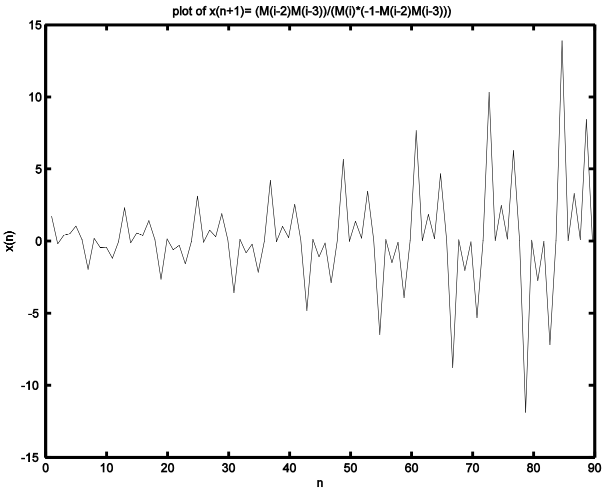

Fig. 7:

<u>Example 8.</u> Fig. 8. shows the solutions when $x_{-3}=5$, $x_{-2}=-2/5$, $x_{-1}=5$, $x_0=-2/5$.

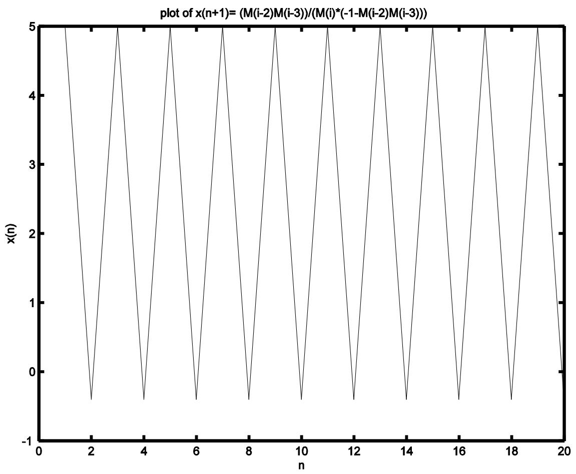

Fig. 8:

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