Australian Journal of Basic and Applied Sciences, 7(12) Oct 2013, Pages: 151-155

AENSI Journals

Australian Journal of Basic and Applied Sciences

Journal home page: www.ajbasweb.com

On Weakly AGP-injective Rings

Raida Dawood Mahmood and Hazem Talat Hazem

Mathematics, College of Computer Sciences And Mathematics, University of Mosul, Iraq

ARTICLE INFO

Article history:
Received 12 September 2013
Received in revised form 17 *November* 2013
Accepted 20 November 2013
Available online 4 December2013

Key words:
AGP-injective, reduced, strongly regular, quasi ZI- ring.

ABSTRACT

In (Stanley and Zhou,1998), introduced a class of AGP-injective rings, following this a ring R is called right AGP-injective if, for any a∈R, there exists n>1 such that $a^n \neq 0$ and Ra^n is a direct summand of $\ell r(a^n)$. In this paper we give a generalization of right AGP-injective rings, we introduced the notion of right weakly AGP-injective rings, that is mean for any maximal right ideal M of R and any a∈M, aR/aM is AGP-injective. Some important results which are known for right AGP-injective rings are shown to holds for right WAGP-injective rings.

To Cite This Article: Raida Dawood Mahmood and Hazem Talat Hazem., On Weakly AGP-injective Rings., ***Aust. J. Basic & Appl. Sci.***, *7(12):151-155, 2013*

INTRODUCTION

Throughout the paper, R will be an associative ring with identity and M is right R-module with S=End (M_R) (Zhao and Xianneng,2012). For a∈R, r(a), ℓ (a) denoted the right annihilator and left annihilator of a, respectively (Abdullah,1998). X≤M denoted that X is a submodule of M (Stanley and Zhao, 1998). We write J(R), Y(R) for the Jacobson radical, the right singular ideal of R (Goodearl, 1979).

As usual, a reduced ring is a ring without nonzero nilpotent elements. If R is reduced, then r (a) = ℓ (a) for any a∈R. R is a right quasi duo (resp. MERT) ring (Xue, 1998), if every maximal (resp., maximal essential) right ideal of R is an ideal. R is regular (Goodearl, 1979), if for every a∈R, there exists b∈R such that a=aba. A ring R is called strongly regular if for every a∈R there exists b∈R such that $a=a^2b$. It is known that a ring R is strongly regular if and only if R is reduced regular ring. A right R-module M is GP-injective ((Yue Chi Ming,1985),(Nam,Rim and Kim,1995)), if for every $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every right R-module, from a^nR to M extends to one from R to M. A ring R is called a right GP-injective ring if R_R is right GP-injective. A right R-module M with S=End (M_R) is said to be AGP-injective (Stanley and Zhou, 1998), if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$, and there exists a left S-submodule X_a of M_R such that $\ell_M r_R(a^n) = Ma^n \oplus X_a$. A right AGP-injective ring need not be right GP-injective [(Stanley and Zhou,1998), Example 1.5]. Actually, many authors investigated some Properties of rings whose simple (simple singular) right R-module is AGP-injective(see (Stanley and Zhou,1998),(Xiao, Nanqing and Wenting, 2004) and (Zhao,Yu-e and Xianneng, 2012).

A ring R is called a ZI-ring (Nam,1999), if for any a,b∈R , ab=0 implies aRb=0. It is well known that R is ZI-ring if and only if r(a) (ℓ (a)) is an ideal of R for every a ∈R. A ring R is abelian if every idempotent of R is central.

2. Weakly AGP-injective Ring:

In this section, we introduce the notion of a right weakly AGP-injective (WAGP-injective) with some of their basic properties also give the relation between such rings and reduced rings, strongly regular rings.

Following, (Abdullah, 1998), a ring R satisfies (*) if for any maximal right ideal M of R and for any a ∈ M, aR/aM is GP-injective.
Now we give the following definition:

Definition(2.1):

A ring R is said to be right (left) weakly almost GP-injective (or for short WAGP-injective) if for any maximal right ideal M of R and for any a∈M, aR/aM (Ra/Ma) is AGP-injective.

Corresponding Author: Raida Dawood Mahmood, Mathematics ,College of Computer Sciences And Mathematics, University of Mosul, Iraq.

Lemma (2.2):

Suppose M is a right R-module with S=End(M_R). If $\ell_M r_R(a^n) = Ma^n \oplus X_a$, where X_a is a left S-submodule of M_R . Set f:$a^nR \longrightarrow M$ is a right R-homomorphism, then f(a^n)=ma^n + x with m∈M , x∈X_a (Zhao, Yu-e and Xianneng, 2012)

The following lemma, which is due to in (Mahmood and Mohammad,(2012) plays a central role in several of our proofs.

Lemma (2.3):

If M is a maximal right ideal of R and r(a)⊆M with a∈M , then

1. aR≠aM.
2. R/M ≃ aR/aM.

We are now in a position to give a few new characteristic properties of reduced rings and strongly regular rings in terms of right weakly almost GP-injective ring.

Proposition(2.4):

Let R a right quasi-duo and right WAGP-injective ring. Then any non zero divisor of R is a right and left invertible.

Proof:

Let a be a non zero divisor of R. If aR≠R , then there exists a maximal right ideal M of R containing aR. Suppose that a R=aM, then a=ab for some b∈M implies 1-b∈r(a)=0, whence 1∈M contradiction M≠ R . If aR≠aM ,then aR/a M ≃R/M .(Lemma(2.3)). Since aR/aM is AGP-injective, then so R/M .Therefore there exists a positive integer n such that $a^n \neq 0$,and $\ell_{R/M} r_R(a^n)$=(R/M)$a^n \oplus X$, $X_{a^n} \leq$R/M . Let f:$a^nR \to R/M$ be defined by $f(a^n r) = r + M$. Since a is a non zero divisor, we have, f a well-defined right R-homorphism.

So f(a^n)=ca^n+M+x ,c∈R, x ∈X . By Lemma (2.2) and f(a^n)=1+M , thus 1- ca^n+M=x∈R/M∩X=0, 1-ca^n ∈M since R quasi duo, ca^n∈M. Which implies 1∈M a contradicting M≠R. Thus aR=R and so az=1 for some z∈R and then we have aza=a implies (1-za)∈r(a)=0,thus za=1.This prove that a is a right and left invertible.

Recall that a right ideal L is called GW-ideal (see (Haiyan,2007)), if for any a∈L, there exists a positive integer n such that Ra^n⊆L. Similarly, the notion of GW-ideal for a left ideal K of R can be defined.

The following theorem extends ((Zhao, Yu-e and Xianneng,2012) ,Theorem (2.9))

Theorem(2.5):

Let R be WAGP-injective and every maximal right ideal of R is a GW-ideal. Then R is reduced.

Proof:

If R is not reduced, then there exist 0≠a∈R such that a^2=0. Hence r(a) is contained in maximal right ideal M of R. Since a^2=0 thus a∈r(a)⊂M. If aR=aM then a=ac for some c∈M, hence 1-c∈r(a)⊂M .Therefore 1∈M, which is a contradiction. Now, if aR≠aM, then aR/aM≃R/M and hence R/M is AGP-injective and $\ell_{R/M}$ r(a)=(R/M)a⊕X. Let f:aR→R/M Lemma (2.3) be defined by f(ar)=r+M.If ar_1=ar_2, then a(r_1-r_2)=0, so (r_1-r_2)∈r(a)⊂M . Hence r_1+M=r_2+M, so f is a well-defined. Since R/M is AGP-injective then there exist b∈R such that, 1+M=f(a)=ba+M+x, x∈X, 1-ba+M=x∈(R/M)∩X= 0, 1-ba∈M. Suppose ba ∉ M, then M+baR=R implying z+bar=1 for some z ∈ M, r ∈ R. Now M is a GW-ideal and arb ∈ M, so there exists a positive integer k such that $b(arb)^k \in M$. Then $(1-z)^{k+1} = (bar)^{k+1} = b(arb)^k ar \in M$ so that 1 ∈ M which is a contradiction. So ba ∈ M and then 1 ∈ M, a gain a contradiction. Thus R is a reduced.

Theorem(2.6):

Let R be a WAGP-injective and every maximal right ideal of R is a GW-ideal .Then R is a strongly regular ring.

Proof:

For every b∈ R we claim that bR+r(b)=R. Suppose that d ∈ R such that dR+r(d)≠ R and let M a maximal right ideal containing dR+r(d). Whence d ∈ M. If dR=dM then d=dc for some c ∈ M. Hence 1-c∈ r(d) ⊂ M therefore 1∈ M which is a contradiction. But if dR ≠dM then dR/dM≃ R/M [Lemma 2.3]. Therefore R/M is AGP-injective and hence there exist a positive integer n such that $d^n \neq 0$ and $\ell_{R/M}$ $r_R(d^n)$=(R/M)d^n ⊕X. Let f:d^nR→R/M be defined by f(d^nr)=r+M. Since R is reduced(Theorem(2.5)) f is well defined R-homorphisem. So f(d^n)=cd^n+M+x, c∈ R, x ∈ X by Lemma (2.2) and f(d^n)=1+M, thus 1-cd^n+M=x∈ (R/M) ∩ X = 0 , 1-cd^n ∈ M.Since M is GW-ideal then applying same methods in the proof of Theorem(2.5). We have 1∈ M which is a gain contradiction. Therefore bR+r(b)=R for all b∈ R. In particular 1=bx+y, y∈r(b), x∈R. So b=b^2x, and therefore R is a strongly regular.

Lemma (2.7) (Goodearl,1979):

If R is regular ring, then every ideal of R is regular ideal.

Following, (Abdullah,1998), an ideal P of a ring R is called completely prime if for every x,y∈R, and xy∈P implies either x∈p or y∈p.

As a parallel result to ((Abdullah,1998), Theorem (2.3.10)) the following result was obtained.

Proposition(2.8):

Let R be a WAGP-injective and every maximal right ideal of R is a GW-ideal then for each completely prime ideal P of R, $P=\bigcup_{a\notin p} r(a)$.

Proof:

Let $Q=\bigcup_{a\notin p} r(a)$ and we want to show that Q=P. Assume that a∉ p, if a∈ Q, then a∈ r(b) for some b∉ P thus ba=0∈ P, and hence a∈ P a contradiction .Therefore Q⊆ P. By Theorem(2.6) R is a strongly regular ring and by Lemma(2.7) P is regular ideal .So for each a∈ P there exists c ∈ P such that a=aca which implies that a(1-ca)=0∈ P.Since P is completely prime ideal of R then 1-ca∉ P for if , then 1∈ P which impossible. Therefore, $a\in \ell(1-ca)=r(1-ca)$ (R is reduced, Theorem(2.5)) thus a∈ Q so that P⊆ Q consequently Q=P.

3.The Regularity of (WAGP)'-Rings:

In this section we give some characteristic properties of quasi ZI-ring, and use quasi ZI-ring to study the connection between generalized simple singular WAGP-injective and strongly regular rings.

Definition (3.1):

A ring R is called right (WAGP)'-ring if every simple singular right R-module is WAGP-injective.

Theorem(3.2):

If R is a (WAGP)'-ring then J(R)∩Y(R) contains no non-zero nilpotent element.

Proof:

Take any b∈ J∩Y with $b^2=0$. If b≠0 then RbR +r(b) is essential right ideal of R. We will show that RbR+r(b)=R. If not, there exists right maximal ideal M of R containing RbR +r(b) .If b R=b M then b=b c for some c∈ M which implies that (1-c)∈r(b)⊆ M when 1∈M contradicting M≠R. If bM ≠bR , then b R/b M≃R/M (Lemma (2.3)). Since bR/bM is AGP-injective then R/M is right AGP-injective and $\ell_{R/M} r_R(b)=(R/M)b \oplus X$. By usual proof mentioned in ((Mahmood and Mohammad,2012), Proposition 2.4) we get a gain contradiction. Thus RbR +r(b)=R, and thus b=ba ,a∈RbR⊆J(R), (1-a) is invertible, then b(1-a)u=b. This implies that b=0, which is a required contradiction.

Definition(3.3) (Xiao, Rui and Xiao,L.2011):

A ring R is called a quasi ZI-ring ,if for a non-zero element a,b∈ R ,ab=0 implies that there exists a positive integer n, such that $a^n \neq 0$ and $a^n R b^n = 0$.

Every a ZI-ring is quasi ZI-ring ,but the converse is not true. ((Xiao, Rui and Xiao,L.2011),Example (2.2)).

Lemma(3.4) (Xiao, Rui and Xiao,L.,2011):

If R is a quasi ZI-ring, then every idempotent element is central in R.

Lemma (3.5):

The following statement are equivalent:

1. R is quasi ZI-ring .
2. For each a∈R, there exists a positive integer n and a^n≠0 such that $\ell(a^n)(r(a^n))$ is a two sided ideal of R.

Proof:

It proved the same method as ((Nam,1999), Lemma 1.2].

Proposition(3.6):

If R is a quasi ZI-ring, right (WAGP)'-ring, then:

1. R is reduced.
2. I+r(a)=R for any non zero ideal I of R and every a∈I.

Proof(1):

Let a∈ R with $a^2 = 0$. If a≠0, then r(a)≠ R and there exists a complement right ideal K of R such that r(a)⊕K is an essential right ideal of R. If r(a)⊕K=R, then r(a)=eR where $e^2 = e \in R$. It follows from Lemma(3.4) that a=ea=ae. Note that a∈ $\ell(r(a)) = \ell(eR)$ =R(1-e), we have a=ae=0. It is a contradiction. If r(a)⊕K≠R, then there exists a maximal right ideal M of R containing r(a)⊕K. Since $a^2 = 0$ thus a∈ $r(a) \subset M$. If aR=aM then a=ad for some d∈M, hence 1-d∈ $r(a) \subset M$ and so $1 \in M$, which is a contradiction. Now if aR≠aM ,then aR/aM≃R/M (Lemma (2.3)) and $\ell_{R/M} r(a) = (R/M)a \oplus X_a$. $X_a \leq R/M$. Let f:aR→ R/M be defined by f(ar)=r+M. Note that f is well define R-homomorphism. Thus f(a)=ba+M+x, b∈R , x∈X, lemma (2.2) and f(a)=1+M, so (1-ba)+M=x∈(R/M) ∩X=0 , we have 1-ba∈ M. Because R is quasi ZI-ring, aRa=0. So aba=0, a=a-aba=a(1-ba)=am, where m=1-ba∈ M. This implies that 1-m∈ $r(a) \subset M$ and $1 \in M$, which is a gain contradiction the maximal of M. So we have a=0 and R is reduced.

Proof(2):

Let I be a non zero ideal of R. If there exists a∈I such that I+r(a)≠R, then there exists right ideal M of R containing I +r(a). We claim that M is an eseential right of R. If not then M is a direct summand of R. Thus M=r(e) for some $0 \neq e^2$=e∈R. Note that a∈r(e). It follows from Lemma (3.4) that ae=ea=0. This implies that e∈r(a)⊆M=r(e), and $e=e^2=0$, which a contradiction. M is an essential right ideal of R. If aR=aM, then a=ad for some d∈ M which implies that (1-d)∈ $r(a) \subset M$ which is a contradiction. If aR≠ aM, then aR/aM≃ (R/M) lemma (2.3). Therefore R/M is AGP-injective, and there exists a positive integer n, such that $a^n \neq 0$ and $\ell_{R/M} r(a^n) = (R/M)a^n \oplus X$. It follows from (1) that $r(a^n) = r(a) \subseteq M$. So we may defined a right R-homomorphism f: $a^n R \to R/M$ by $f(a^n r) = r + M$ for all $r \in R$. Then there exists $c \in R$ such that 1+M=f(a^n)=ca^n +M+x, x∈ X [Lemma 2.2], $1 - ca^n + M = x \in R/M \cap X = 0$,$1 - ca^n \in M$. Because ca^n∈ I ⊆M, we have 1∈ M, which is a gain contradiction .Therefore I+r(a)=R, for any non zero I of R and every a ∈ I.

As a parallel result ((Mahmood and Mohammad,2012), Proposition 2.7], the following result was obtained.

Proposition (3.7):

Let R be a quasi ZI, (WAGP)'-rings, then for any a,b∈R, with ab=0, then r(a)+r(b)=R.

Proof:

Suppose that ab=0, and r(a)+r(b)≠R. Then there exists a maximal right ideal M containing r(a)+r(b). If M is not essential, then there exists $0 \neq e = e^2 \in R$ such that M=r(e) since b∈r(a)⊆M=r(e)=ℓ(e) Lemma (3.4) then be=0 which implies that e∈r(b) ⊆M=r(e), so that $e=e^2=0$ which is a contradiction. Therefore M must be essential. If aR=aM, then a=ab for some b∈M which implies (1-b) ∈r(a) ⊆M, which implies 1∈M, a contradiction. If aR≠aM , then aR/aM≃R/M Lemma (2.3). Since aR/aM is AGP-injective then R/M is AGP-injective. Let f:a^nR→R/M is defined by f(a^nr)=r+M, for all r∈R, n is a positive integer such that $a^n \neq 0$. Note that f is well defined (R is reduced) and by Lemma (2.2) 1+M=f(a^n)=ca^n+M+x, c ∈ R, x ∈ X. Hence 1-ca^n∈M=x∈(R/M)∩X=0 so 1-ca^n∈M. Since a∈r(b) and R is quasi ZI, then ca^n∈r(b) ⊆M, whence 1∈M which is a contradiction. Therefore r(a)+r(b)=R.

Now, we give some new characterization of strongly regular rings as follows:

Theorem(3.8):

If R is a quasi ZI-rings, then the following statement are equivalent:

1. R is strongly regular ring.
2. R is an MELT, left WAGP-injective ring .
3. R is an MERT, right WAGP-injective ring.
4. R is an MELT, left (WAGP)'-ring.
5. R is an MERT, right (WAGP)'- ring.

Proof:

Obviously 1→3→5, and 1→2→4. 4→1 and 5→1 is similar. We need only to prove 5→1.Suppose (5),for any $0 \neq a \in R$,we will show that aR+r(a)=R . If not, then there exists a maximal right ideal M of R containing aR+r(a).with the similar discussion to the proof of Proposition (3.6), we get that M is an essential right ideal of R. If aR=aM, then a=ac for some $c \in M$ hence 1-c∈ $r(a) \subseteq M$. So $1 \in M$, a contradiction. If aR≠ aM, then $aR/aM \simeq R/M$ and hence R/M is AGP-injective. Thus there exists a positive integer n such that $a^n \neq 0$ and $\ell_{R/M} r(a^n) = (R/M)a^n \oplus X$. Let f:$a^n$R→R/M be defined by f($a^n$r)=r+M. Since R is reduced f is well defined and f(a^n)=1+M and so 1-ca^n+M=x∈(R/M)∩X=0, $1 - ca^n \in M$, (c∈ R), since R is MERT, M is an ideal of R and RaR⊆ M, which implies that $1 \in M$. This is a gain contradiction. Therefore, there exists b∈ R, d∈r(a) such that 1=ab+d, and a=a^2b. So R is of strongly regularity.

REFERENCES

Abdullah, A.M., 1998. "On YJ-injective Modules", M.S.C. Theses, Mosel University.

Goodearl, K.R., 1979. "Von-Neumann Regular Rings" pitman (London).

Haiyan, Z., 2007. "Left SF-rings and regular rings", Comm. Algebra, 35: 3842-3850. 1, 1009-1018.

Mahmood, R.D. and H.Q. Mohammad, 2012. "On generalized simple singular AP-injective rings", Rafidain J.of Computer Science Math., 9(1).

Nam, S.B., N.K. Rim and J.Y. Kim, 1995, On simple GP-injective modules". Algebra, 23(14): 5437-5444.

Nam, S.B., 1999. "A not on simple singular GP-injective modules", Kyu. Math. J., 7(2): 215-218.

Stanley, S.P. and Y. Zhou, 1998. "Generalizations of principally injective rings", Journal of Algebra, 206: 706-721.

Xue, W., 1998. "A note on YJ-injective", Riv. Math.Univ. Parma, 1(6): 31-37.

Xiao, B.Y. Yin, W. Rui and L. Xiao, 2011. "The non singularity and regularity of GP-V̀ –rings", J. of Math. Reserach and Exposition, 31(6): 1003-1008.

Xiao, G., D. Nanqing and T. Wenting, 2004. "Regularity of AGP-injective rings", Vietnam. J. Math., 32(4): 399-411.

Yue Chi Ming, R., 1985. "On regular rings and artinian rings II", Riv. Math. Unive. Parma, 4(11): 101-109.

Zhao, Yu-e and D. Xianneng, 2012. "On simple AGP-injective module ", Internatinal. J. of Algebra, 6(9): 441-448.