On Weakly AGP-injective Rings

Raida Dawood Mahmood and Hazem Talat Hazem

Mathematics, College of Computer Sciences And Mathematics, University of Mosul, Iraq

ABSTRACT

In (Stanley and Zhou,1998), introduced a class of AGP-injective rings, following this a ring R is called right AGP-injective if, for any a∈R, there exists n>1 such that a^n=0 and Ra^n is a direct summand of r(a^n). In this paper we give a generalization of right AGP-injective rings, we introduced the notion of right weakly AGP-injective rings, that is mean for any maximal right ideal M of R and any a∈M, aRa/M(M) is AGP-injective. Some important results which are known for right AGP-injective rings are shown to holds for right WAGP-injective rings.

INTRODUCTION

Throughout the paper, R will be an associative ring with identity and M is right R-module with S=End (M_R) (Zhao and Xianeng,2012). For a∈R, r(a), l(a) denoted the right annihilator and left annihilator of a, respectively (Abdullah,1998). X≤M denoted that X is a submodule of M (Stanley and Zhao, 1998). We write J(R), Y(R) for the Jacobson radical, the right singular ideal of R (Goodearl, 1979).

As usual, a reduced ring is a ring without nonzero nilpotent elements. If R is reduced, then r (a) = l (a) for any a∈R. R is a right quasi duo (resp. MERT) ring (Xue, 1998), if every maximal (resp., maximal essential) right ideal of R is an ideal. R is regular (Goodearl, 1979), if for every a∈R, there exists b∈R such that a=aba. A ring R is called strongly regular if for every a∈R there exists b∈R such that a=a'a. It is known that a ring R is strongly regular if and only if R is reduced regular ring. A right R-module M is GP-injective ([Yue Chi Ming,1985],(Nam,Rim and Kim,1995)), if for every 0 ≠a∈M, there exists a positive integer n such that a^n≠0 and every right R-module, from a^R to M extends to one from R to M. A ring R is called a right GP-injective ring if R is right GP-injective. A right R-module M with S=End (M_R) is said to be AGP-injective (Stanley and Zhou, 1998), if for any 0 ≠a∈M, there exists a positive integer n such that a^n≠0, and there exists a left S-submodule X of M such that (a^n) = Ma^n ⊕ X. A right AGP-injective ring need not be right GP-injective ([Stanley and Zhou,1998], Example 1.5). Actually, many authors investigated some Properties of rings whose simple (simple singular) right R-module is AGP-injective(see (Stanley and Zhou,1998),(Xiao, Nanqing and Wenting, 2004) and (Zhao,Yu-e and Xianeng, 2012).

A ring R is called a ZI-ring (Nam,1999), if for any a,b∈R , ab=0 implies aRb=0. It is well known that R is ZI-ring if and only if r(a) (l(a)) is an ideal of R for every a∈R. A ring R is abelian if every idempotent of R is central.

2. Weakly AGP-injective Ring:

In this section, we introduce the notion of a right weakly AGP-injective (WAGP-injective) with some of their basic properties also give the relation between such rings and reduced rings, strongly regular rings.

Following, (Abdullah, 1998), a ring R satisfies (*) if for any maximal right ideal M of R and for any a ∈ M, aRa/M(M) is GP-injective.

Now we give the following definition:

Definition(2.1):

A ring R is said to be right (left) weakly almost GP-injective (or for short WAGP-injective) if for any maximal right ideal M of R and for any a∈M, aRa/M(M) is AGP-injective.
Lemma (2.2): 
Suppose M is a right R-module with S=End(M_R). If f_Mr(a^n) = Ma^n ⊕ X_a, where X_a is a left S-submodule of M_R. Set f:aR→M is a right R-homomorphism, then f(a^n)=ma^n + x with m∈M, x∈X_a (Zhao, Yu-e and Xianeng, 2012).

The following lemma, which is due to in (Mahmood and Mohammad, 2012) plays a central role in several of our proofs.

Lemma (2.3): 
If M is a maximal right ideal of R and r(a)⊆M with a∈M, then 
1. aR# aM.
2. R/M = aR/aM.

We are now in a position to give a few new characteristic properties of reduced rings and strongly regular rings in terms of right weakly almost GP-injective ring.

Proposition(2.4): 
Let R a right quasi-duo and right WAGP-injective ring. Then any non zero divisor of R is a right and left invertible.

Proof: 
Let a be a non zero divisor of R. If aR∩R , then there exists a maximal right ideal M of R containing aR. Suppose that aR=aM, then a=ab for some b∈M implies 1-b=a=0, whence 1M contradiction M# R . If aR# aM , then ar(a)=0,(Lemma(2.3)). Since aR/aM is AGP-injective, then so R/M .Therefore there exists a positive integer n such that a^n ≠ 0, and (R/M)a^n ⊕ X, X_a ⊆ R/M . Let f:a^nR→R/M be defined by f(a^n)= r + M. Since a is a non zero divisor, we have, f a well-defined right R-homomorphism.

So f(a^n)=ca^n+M+x ,c∈R, x∈X . By Lemma (2.2) and f(a^n)=1+M , thus 1- ca^n+M=x∈R/M∩X=0, 1-ca^n EM since R quasi duo, ca∈M. Which implies 1EM a contracting M#R. Thus aR=R and so az=1 for some z∈R and then we have az=az implies (1-za)∈R(a)=0,thus za=1. This prove that a is a right and left invertible.

Recall that a right ideal L is called GW-ideal (see (Haiyan, 2007)). If for any a∈L, there exists a positive integer n such that Ra⊆L. Similarly, the notion of GW-ideal for a left ideal K of R can be defined.

The following theorem extends ((Zhao, Yu-e and Xianeng, 2012),Theorem (2.9))

Theorem(2.5): 
Let R be WAGP-injective and every maximal right ideal of R is a GW-ideal. Then R is reduced.

Proof: 
If R is not reduced, then there exist 0≠a∈R such that a^2=0. Hence r(a) is contained in maximal right ideal M of R. Since a^2=0 thus aR∩R⊆M. If aR=aM then a=ac for some c∈M, hence 1-c∈R(a)⊆M .Therefore 1∈M, which is a contradiction. Now, if aR# aM, then aR/aM ≃ R/M and hence R/M is AGP-injective and f_R/M r(a)=R/MaQX. Let f:aR→R/M Lemma (2.3) be defined by f(ar)=r+aM. If ar=aR, then ar=(aR)R=0, so (R/ r_aR∈R(a)⊆M . Hence r_aR=MaQ+M, so f is well-defined. Since R/M is AGP-injective then there exist b∈R such that, 1M=f(a)=ba+aM+x, x∈X . 1-baM=x∈R/M∩X=0, 1-baM. Suppose ba∈M , then M+baR=R implying x+bar=1 for some z∈M, r∈R. Now M is a GW-ideal and arb∈R, so there exists a positive integer k such that b(ar)^k∈M. Then (1-z)^k+1 = (bar)^k+1 = b(ar)k+1 ∈ M so that z∈M which is a contradiction. So ba∈M and then 1∈M, a gain a contraction. Thus R is a reduced.

Theorem(2.6): 
Let R be a WAGP-injective and every maximal right ideal of R is a GW-ideal. Then R is a strongly regular ring.

Proof: 
For every b∈R we claim that bR+r(b)=R. Suppose that d∈R such that dR+r(d)≠ R and let M a maximal right ideal containing dR+r(d). Whence d ∈ M. If dR=dM then d=dc for some c∈M. Hence 1-c∈ r(d)⊆ M therefore 1∈M which is a contradiction. But if dR ≠ dM then dR/dM = R/M [Lemma 2.3]. Therefore R/M is AGP-injective and hence there exist a positive integer n such that d^n ≠ 0 and f_R/M r(d^n)=R/Ma^n ⊕ X. Let f:d^nR→R/M be defined by f(d^n)r=1+M. Since R is reduced(Theorem(2.5)) f is well-defined R-homorphisms. So f(d^n)=c+d^n+M+x, c∈R, x∈X by Lemma (2.2) and f(d^n)=1+M, thus 1-cd^n+M=x∈R/M∩X=0 , 1-cd^n∈M. Since M is GW-ideal then applying same methods in the proof of Theorem(2.5). We have 1∈M which is a gain contradiction. Therefore bR+r(b)=R for all b∈R. In particular 1=bx+y, y∈R(b), x∈R. So b=b^2x, and therefore R is a strongly regular.
Lemma (2.7) (Goodearl, 1979):
If $R$ is a regular ring, then every ideal of $R$ is a regular ideal.
Following, (Abdullah, 1998), an ideal $P$ of a ring $R$ is called completely prime if for every $x, y \in R$, and $xy \in P$ implies either $x \in P$ or $y \in P$.
As a parallel result to ((Abdullah, 1998), Theorem (2.3.10)) the following result was obtained.

Proposition (2.8):
Let $R$ be a WAGP-injective and every maximal right ideal of $R$ is a GW-ideal then for each completely prime ideal $P$ of $R$, $P = \bigcup_{a \notin P} r(a)$.

Proof:
Let $Q = \bigcup_{a \notin P} r(a)$ and we want to show that $Q = P$. Assume that $a \notin P$, then $a \notin r(b)$ for some $b \notin P$ and hence $aeP$ a contradiction. Therefore $Q \subseteq P$. By Theorem (2.6) $R$ is a strongly regular ring and by Lemma (2.7) $P$ is a regular ideal. So for each $a \in P$ there exists $c \in P$ such that $a =aca$ which implies that $a(1-ca) = 0 \in P$. Since $P$ is completely prime ideal of $R$ then $1-ca \notin P$ for if, then $1 \in P$ which impossible. Therefore, $a \in (1-ca) = r(1-ca)$ (R is reduced, Theorem (2.5)) thus $a \in Q$ so that $P \subseteq Q$ consequently $Q = P$.

3. The Regularity of (WAGP)'-Rings:
In this section we give some characteristic properties of quasi ZI-ring, and use quasi ZI-ring to study the connection between generalized simple singular WAGP-injective and strongly regular rings.

Definition (3.1):
A ring $R$ is called right (WAGP)'-ring if every simple singular right $R$-module is WAGP-injective.

Theorem (3.2):
If $R$ is a (WAGP)'-ring then $J(R) \cap Y(R)$ contains no non-zero nilpotent element.

Proof:
Take any $b \in J \cap Y$ with $b^2 = 0$. If $b \neq 0$ then $RbR + r(b)$ is essential right ideal of $R$. We will show that $RbR + r(b) = R$. If not, there exists right maximal ideal $M$ of $R$ containing $RbR + r(b)$ and $b \in M$ which implies that $b = (1-ca) \in M$ when $1-ca \notin M$ contradicting $M \neq R$. If $bM \neq R$, then $bR/bM \cong (R/M)b$ (Lemma (2.3)). Since $bR/bM$ is AGP-injective then $R/M$ is right AGP-injective and $\ell(R/M)b = (R/M)b \oplus X$. By usual proof mentioned in ((Mahmood and Mohammad, 2012), Proposition 2.4) we get a gain contradiction. Thus $RbR + r(b) = R$, and thus $b = ba$, $a \in RbR \subseteq J(R)$, $(1-a)$ is invertible, then $b(1-a)u = b$. This implies that $b = 0$, which is a required contradiction.

Definition (3.3) (Xiao, Rui and Xiao, L., 2011):
A ring $R$ is called quasi ZI-ring if for a non-zero element $a, b \in R$, $ab = 0$ implies that there exists a positive integer $n$, such that $a^n \neq 0$ and $a^nRb^n = 0$.
Every $a$-ZI ring is quasi ZI-ring, but the converse is not true. ((Xiao, Rui and Xiao, L., 2011), Example (2.2)).

Lemma (3.4) (Xiao, Rui and Xiao, L., 2011):
If $R$ is a quasi ZI-ring, then every idempotent element is central in $R$.

Lemma (3.5):
The following statement are equivalent:
1. $R$ is quasi ZI-ring.
2. For each $a \in R$, there exists a positive integer $n$ and $a^n \neq 0$ such that $(a^n)(r(a^n))$ is a two sided ideal of $R$.

Proof:
It proved the same method as ((Nam, 1999), Lemma 1.2).

Proposition (3.6):
If $R$ is a quasi ZI-ring, right (WAGP)'-ring, then:
1. $R$ is reduced.
2. $1+r(a) = R$ for any non-zero ideal I of $R$ and every $a \in I$. 
Proof(1):

Let \( a \in R \) with \( a^2 = 0 \). If \( a \neq 0 \), then \( r(a) \neq R \) and there exists a complement right ideal \( K \) of \( R \) such that \( r(a) \oplus K \) is an essential right ideal of \( R \). If \( r(a) \oplus K = R \), then \( r(a) = eR \) where \( e^2 = e \in R \). It follows from Lemma (3.4) that \( a = eae = 0 \). Note that \( a \notin (r(a)) = (eR) = R(1-e) \), we have \( a = ae = 0 \). It is a contradiction. If \( r(a) \oplus K = R \), then there exists a maximal right ideal \( M \) of \( R \) containing \( r(a) \oplus K \). Since \( a^2 = 0 \) thus \( a \notin r(a) \subseteq M \). If \( aR \cong M \) then \( aR = aM \) for some \( e \in M \), hence \( 1-\alpha \in M \) and so \( 1 \in M \), which is a contradiction. Now if \( aR \neq aM \), then \( aR/aM \cong R/M \) (Lemma (2.3)) and \( f_{R/M}(a) = (R/M)a \oplus X \). Let \( f : aR \rightarrow R/M \) be defined by \( f(a) = rM \). Note that \( f \) is well defined \( R \)-homomorphism. Thus \( f(a) = ba + M + x \), \( b \in R \), \( x \in X \), lemma (2.2) and \( f(a) = 1+M + x \in (R/M) \cap X = 0 \), we have \( 1-ba = M \). Because \( R \) is quasi ZI, \( aR = 0 \). So \( ab = 0 \). Therefore \( a = a-ba = a(1-ba) = m \), then \( m = 1-ba \in M \). This implies that \( 1-m \in r(a) \subseteq M \) and \( 1, M \), which is a gain contradiction the maximal of \( M \). So we have \( a = 0 \) and \( R \) is reduced.

Proof(2):

Let \( I \) be a non zero ideal of \( R \). If there exists \( a \in I \) such that \( 1+r(a) \neq R \), then there exists right ideal \( M \) of \( R \) containing \( 1 + r(a) \). We claim that \( M \) is an essential right of \( R \). If not then \( M \) is a direct summand of \( R \). Thus \( M = r(e) \) for some \( 0 < e \leq eR \). Note that \( e \in r(e) \). It follows from Lemma (3.4) that \( eae = 0 \). This implies that \( e \in r(a) \subseteq M = r(e) \), which is a contradiction. \( M \) is an essential right ideal of \( R \). If \( aR = aM \), then \( aR/aM \cong R/M \) (Lemma (2.3)). Therefore \( R/M \) is AGP-injective, and there exists a positive integer \( n \), such that \( a^n \neq 0 \) and \( f_{R/M}(a^n) = (R/M)a^n \oplus X \). It follows from (1) that \( a^n \in R \subseteq M \). So we may define a \( R \)-homomorphism \( f: a^n \rightarrow R/M \) by \( f(a^n) = r + M \) for all \( r \in R \). Then there exists \( c \in R \) such that \( 1+aM = f(a^n) = ca^n + M + x \), \( x \in X \) [Lemma 2.2]. \( 1 - ca^n + M = x \in R/M \cap X = 0 \), \( 1 - cM \in M \). Because \( ca^n \in I \subseteq M \), we have \( 1 \in M \), which is a gain contradiction. Therefore \( 1+r(a) = R \), for any non zero \( I \) of \( R \) and every \( a \in I \).

As a parallel result ((Mahmood and Mohammad, 2012), Proposition 2.7), the following result was obtained.

Proposition (3.7):

Let \( R \) be a quasi ZI, \( (WAGP)^\dagger \)-rings, then for any \( a, b \in R \), with \( ab = 0 \), then \( r(a)+r(b) = R \).

Proof:

Suppose that \( ab = 0 \), and \( r(a)+r(b) \neq R \). Then there exists a maximal right ideal \( M \) containing \( r(a)+r(b) \). If \( M \) is not essential, then there exists \( 0 < e < e \in R \) such that \( M = r(e) \) since \( b \in r(a) \subseteq M = r(e) \subseteq M \). Lemma (3.4) then \( be = 0 \) which implies that \( e \in r(b) \subseteq M = r(e) \), so that \( e = e^2 = 0 \) which is a contradiction. Therefore \( M \) must be essential. If \( aR = aM \), then \( a = ab \) for some \( b \in M \) which implies \( 1-b \in r(a) \subseteq M \), which implies \( 1 \in M \), a contradiction. If \( aR \neq aM \), then \( aR/aM \cong R/M \) (Lemma (2.3)). Since \( aR/aM \) is AGP-injective then \( R/M \) is AGP-injective. Let \( f : aR \rightarrow R/M \) be defined by \( f(a^n) = r + M \), for all \( r \in R \), \( n \) is a positive integer such that \( a^n \neq 0 \). Note that \( f \) is well defined \( R \) is reduced by Lemma (2.2) \( 1+aM = f(a^n) = ca^n + M + x \), \( c \in R \), \( x \in X \). Hence \( 1-ca^n \in M = x \in R/M \cap X = 0 \), so \( 1-ca^n \in R \). Since \( ab = 0 \) and \( R \) is quasi ZI, \( ca^n \in M \), whence \( 1 \in M \) which is a contradiction. Therefore \( r(a)+r(b) = R \).

Now, we give some new characterization of strongly regular rings as follows:

Theorem (3.8):

If \( R \) is a quasi ZI-rings, then the following statement are equivalent:
1. \( R \) is strongly regular ring.
2. \( R \) is an MELT, left WAGP-injective ring.
3. \( R \) is an MERT, right WAGP-injective ring.
4. \( R \) is an MELT, left \( (WAGP)^\dagger \)-ring.
5. \( R \) is an MERT, right \( (WAGP)^\dagger \)-ring.

Proof:

Obviously \( 1 \rightarrow 3 \rightarrow 5 \), and \( 1 \rightarrow 2 \rightarrow 4 \), and \( 2 \rightarrow 1 \rightarrow 5 \). Suppose (5), for any \( a \in R \), we will show that \( aR + r(a) = R \). If not, then there exists a maximal right ideal \( M \) of \( R \) containing \( aR + r(a) \), with the similar discussion to the proof of Proposition (3.6), we get that \( M \) is an essential right ideal of \( R \). If \( aR = aM \), then \( a = ac \) for some \( c \in M \) and hence \( 1-ae \in R \subseteq M \). So \( 1 \in M \), a contradiction. If \( aR \neq aM \), then \( aR/aM \cong R/M \) and hence \( R/M \) is AGP-injective. Thus there exists a positive integer \( n \) such that \( a^n \neq 0 \) and \( f_{R/M}(a^n) = (R/M)a^n \oplus X \). Let \( f : aR \rightarrow R/M \) be defined by \( f(a^n) = r + M \). Since \( R \) is reduced \( f \) is well defined and \( f(a^n) = 1 + M \) and so \( 1-ca^n \in X \in (R/M) \cap X = 0 \), \( 1 \notin M \), \( c \in R \), since \( R \) is MERT, \( M \) is an ideal of \( R \) and \( Ra \subseteq M \), which implies that \( 1 \in M \). This is a gain contradiction. Therefore, there exists \( b \in R \), \( d \in (r(a)) \) such that \( 1 = ab + d \), and \( a = a^2 b \). So \( R \) is of strongly regularity.
REFERENCES