On Generalized Minimal-Open Set and Some Properties

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Abstract

In this paper we introduced two topological concepts ,minimal regular open(briefly,m_r(X)) set and minimal pre-regular open set(briefly,m_r(PR(X))) and the concept minimal open(briefly,m_r(U(X))) set comes between them and we studied the relations between these two concepts with the concept minimal α-open set(briefly,m_α(U(X))) . In addition to that we proved some of their proposition .At last we introduced other concepts like generalized’ regular minimal closed set(briefly, g r-closed), generalised’ minimal closed set (briefly,rg-closed), generalized’ regular minimal closed set (briefly, g r-mcg(X)) andgeneralized’ minimal closed set(briefly,rg-gmcg(X)),we explained the relations between these concepts and generalized minimal closed set and gave some important examples and some theorems.

Introduction

The concept of minimal open set was introduced first by Nakaoaka. AndOda (2001,2003,2003). The concept of generalized closed set was first introduced by Levine(1970). In this paper we shall introduce , the concept of regular generalized’ minimal-closed set is to be introduced which lies between generalized’ regular minimal-closed sets and regular generalized’ minimal-closed sets, and introduce the generalized” regular minimal-closed set and some properties of these sets are studies and the corresponding topological space. Moreover It can be shown that the relation with regular generalized’ minimal-closed set and generalized” regular minimal-closed set are independent and some properties and characteristics of these would be given.

2. Preliminaries:

Some important preliminaries required to go further through this paper are cited below.

Definition 2.1:

A subset A of a space X is called a generalized closed set (briefly, g - closed ) (Bhattacharya, S., Halder,2011) if CIA(U) whenever A ≤U and U is an open set.

Definition 2.2:

A subset A of a space X is called a regular generalized closed (briefly, rg- closed) set(Vadival, A. and Variya Manickam, K., 2009) if CIA(U) whenever A ≤U and U is a regular open set.

Definition 2.3:

A subset A of a space Xis called a generalized regular closed[ resp. a generalized regular open] set (briefly, gr- closed)[ resp briefy,gr- open](Bhattacharya, S., Halder,2011) if RCl(A) ≤U [respU ≤R Int(A)] whenever A ≤U [ respU ≤A] , and U is an open [resp. a closed] subset of X.

Example 2.4:

Let X = {a, b, c}/(Bhattacharya, S., Halder,2011)and the corresponding topological space be T = {φ, X, {a}, {b,c}}. Let A = {b}. Here A is a generalized regular closed set of X. Though it is not a regular closed subset of X. Similarly it can be shown that, /b, c/ is a generalized regular open subset of X.
**Definition 2.5:**
A proper nonempty open subset $U$ of a topological space $X$ is called to be a minimal open set (Nakaoka, F. and Oda, N., 2003) if any open set which is contained in $U$ is $\emptyset$ or $U$.

**Definition 2.6:**
A proper nonempty open subset $U$ of a topological space $X$ is called to be maximal open set (Nakaoka, F. and Oda, N., 2003) if any open set which contains $U$ is $X$ or $U$.

**Definition 2.7:**
A proper nonempty closed subset $F$ of a topological space $X$ is called to be a minimal closed set (Nakaoka, F. and Oda, N., 2003) if any closed set which contains $F$ is $X$ or $F$.

**Definition 2.8:**
A proper nonempty closed subset $F$ of a topological space $X$ is called to be maximal closed set (Nakaoka, F. and Oda, N., 2003) if any closed set which contains $F$ is $X$ or $F$.

**Definition 2.9:**
A subset $A$ of $X$ is called to be a generalized minimal closed (resp. generalized maximal open) set (Bhattacharya, S., Halder, 2011) if $A$ is contained [resp. $A$ contains] in at least one minimal open [resp. at least one maximal closed] subset $U$ of $X$ such that $\text{Cl}(A) \supseteq U$ [resp. $\text{Int}(A) \subseteq U$]

**Example 2.10:**
Let $X=\{a,b,c,d\}$ (Bhattacharya, S., Halder, 2011) and the topology be $\mathcal{T} = \{\emptyset, \{b,c\}, \{a\}, \{a,b,c\}, X\}$. Let $A = \{c\} \subseteq \{b,c\}$. Then $\text{Cl}(A) = \{b,c,d\} \supseteq \{b,c\}$ $\Rightarrow$ $A$ is a generalized minimal closed.

**Definition 2.11:**
Let $A$ be a subset of a space $X$ the closure of $A$ (Lipschutz S., 1965), denoted by $\text{Cl}(A)$ is the intersection of all closed supersets of $A$. In other words, if $\{F_i : i \in I\}$ is the class of all closed subsets of $X$ containing $A$ then $\text{Cl}(A) = \cap F_i$.

**Definition 2.12:**
A subset $A$ of a space $X$ is called
1) an pre-open set (Mashhour, A.S., Abd El-Monsef, M.E. and El-Deeb, S.N., 1982) if $A \subseteq \text{Int}(\text{cl}(A))$ and a pre closed set if $\text{Cl}(\text{Int}(A)) \subseteq A$;
2) an $\alpha$-open set (Njastad, O., 1965) if $A \subseteq \text{Int}(\text{cl}(\text{Int}(A)))$ and $\alpha$-closed set if $\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A$;
3) an regular open set (Palaniappan, N. and Rao, K.C., 1993) if $A = \text{Int}(\text{Cl}(A))$ and a regular closed set if $A = \text{Cl}(\text{Int}(A))$.

**Definition 2.13:**
Let $A$ be a subset of a space $X$ the interior of $A$ (Lipschutz S., 1965), denoted by $\text{Int}(A)$ is the union of all open subsets of $X$ contained in $A$ In other words, if $\{F_i : i \in I\}$ is the class of all open subsets of $X$ containing $A$ then $\text{Int}(A) = \bigcup F_i$.

**Definition 2.14:**
A subset $A$ of a space $X$ is called a generalized regular closed set [resp. generalized regular open set] (briefly $g^r$-closed set, $g^r$-open set) (Bhattacharya, S., Halder, 2011) if $\text{RCl}(A) \supseteq U$ [resp. $\text{RInt}(A) \subseteq U$] whenever $A \subseteq U$ [resp. $U \subseteq A$] and $U$ is a regular open [resp. regular closed] subset of $X$.

**Example 2.15:**
Let $X = \{a,b,c\}$ (Bhattacharya, S., Halder, 2011) and the corresponding topological space be $\mathcal{T} = \{\emptyset, \{a\}, \{a,b\}, X\}$
Here the only regular open sets are $X$ and $\emptyset$
Let $A = \{a\}$. Clearly $A \subseteq X$, the regular open set $\text{RCl}(A) = X$, which is also subset of $X$. So $A$ is a generalized regular closed set.

**Definition 2.16:**
Let $X$ be a space and $A \subseteq X$ an $\alpha$-open set. Then $A$ is called a minimal $\alpha$-open set (Mohammed, M., Nokhas, 2013) if $\emptyset$ and $A$ are the only $\alpha$-open subsets of $A$. 


Theorem 2.17:
A subset $A$ of $X$ is a generalized $^*$ minimal closed (Bhattacharya, S., Halder, 2011) iff there exist a minimal open set $U$ containing $A$ such that $\text{Cl}(A) = \text{Cl}(U)$

Remark 2.18:
$\emptyset$ and $X$ are not generalized $^*$ minimal closed (Bhattacharya, S., Halder, 2011).

Theorem 2.19:
Arbitrary union of generalized $^*$ minimal closed set (Bhattacharya, S., Halder, 2011) is a generalized $^*$ minimal closed set if it is contained in a minimal open set.

Remark 2.20:
Finite intersection of generalized $^*$ minimal closed set (Bhattacharya, S., Halder, 2011) need not be a generalized $^*$ minimal closed set which follows example.

Let $X = \{a, b, c\}$ and the corresponding topological space be $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{c\}, X\}$. Let $A = \{c\}$ be a subset of $X$. Obviously $A$ is a generalized $^*$ minimal closed set. Let $B = \{a\}$ be another subset of $X$. $B$ is also a generalized $^*$ minimal closed set of $X$. But $A \cap B = \emptyset$ is not a generalized $^*$ minimal closed set of $X$.

Theorem 2.21:
Non–null intersection of a generalized $^*$ minimal closed set (Bhattacharya, S., Halder, 2011) and a closed is a generalized minimal closed set.

Theorem 2.22:
Let $A$ be a generalized $^*$ minimal closed set (Bhattacharya, S., Halder, 2011) and $B$ be a subset of $X$ contained in the same minimal open set if $A \subseteq B \subseteq \text{Cl}(A)$, then $B$ is also a generalized minimal closed set.

3- On generalized minimal-open set and some properties:
Now we introduced a new concepts in a topological space and some of their properties with many examples.

Definition 3.1:
Let $X$ be a space and $A \subseteq X$ an regular open set then $A$ is called a minimal regular open set if $\emptyset$ and $A$ are only regular open subsets of $A$. The family of all minimal regular open set is denoted by $m_{1-RO}(X)$.

Example 3.2:
Let $X = \{a, b, c\}$ and the topology be $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{b\}, X\}$. $\text{RO}(X) = \{\emptyset, \{a\}, \{b\}, X\}$. $A = \{a\}$ is $m_{1-B-open}$ set $\Rightarrow A$ is minimal regular open set.

Definition 3.3:
Let $X$ be a space and $A \subseteq X$ an regular closed set then $A$ is called a maximal regular closed set if $\emptyset$ and $A$ are only regular closed subsets of $A$. The family of all maximal regular closed set is denoted by $m_{a-RC}(X)$.

Recall Example 3.2.
$A = \{b, c\}$ is $m_{a-k-closed}$ set.

Definition 3.4:
Let $X$ be a space and $A \subseteq X$ an pre regular open set then $A$ is called a minimal pre regular open set if $\emptyset$ and $A$ are only pre regular open subsets of $A$. The family of all minimal pre regular open set is denoted by $m_{1-PR-O}(X)$.

Recall Example 3.2.
$A = \{b\}$ is $m_{1-PR-open}$ set.

Example 3.5:
$\text{PR}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
$\text{PRc}(X) = \{\{b, c\}, \{a, c\}, \{c\}, \emptyset\}$
$A = \{b\}$ is $m_{1-PR-open}$ set.
Definition 3.6:
Let $X$ be a space and $A \subseteq X$ an pre regular closed set then $A$ is called a maximal pre regular closed set if $\emptyset$ and $A$ are only pre regular closed subsets of $A$. The family of all maximal pre regular closed set is denoted by $m_{1-prc}(X)$

Recall example 3.5
$A=\{c\}$ is $m_{a-pr-closed}$ set

Now we introduce some relations among these definitions as

Proposition 3.7:
Every minimal regular open set is minimal pre regular open set.
but the converse is not true.

Proof:
Let $A$ is a minimal regular open set. To prove $A$ is a minimal pre regular open set.
Since $A$ is regular open set and every regular open set is pre regular open set.
Then minimal regular open set is minimal pre regular open set.
Therefore $A$ is a minimal pre regular open set

Example 3.8:
Let $X=\{a,b,c\}$ and the topology be $T=\{\emptyset,\{a\},\{c\},\{a,c\},\{b,c\},X\}$
$PRo(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},X\}$
$A=\{c\}$ is $m_{1-pr-closed}$ set but not $m_{1-R-open}$ set

Proposition 3.9:
Every minimal $\alpha$–open set is minimal pre regular open set.
but the converse is not true.

Proof:
Let $A$ is a minimal $\alpha$–open set. To prove $A$ is a minimal pre regular open set.
Since $A$ is $m_{1-\alpha-open}$ set
Since every $\alpha$-open set is Pre-open set And every $\alpha$-open set is PR-open set
Then $m_{1-\alpha-open}$ set is $m_{1-pr-open}$ set
Therefore $A$ is a minimal pre regular open set

Example 3.10:
Let $X=\{a,b,c,d\}$ and the topology be $T=\{\emptyset,\{a\},\{c,d\},\{a,c,d\},x\}$
$PRo(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{b,c,d\},X\}$
$AO(X)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\{b,c,d\},X\}$
$A=\{a\}$ is $m_{1-pr-open}$ set but not $m_{1-\alpha-open}$ set.

Proposition 3.11:
Every minimal regular open set minimal $\alpha$–open set. but the converse is not true.

Proof:
Let $A$ is a minimal regular open set. To prove $A$ is a minimal $\alpha$–open set
Since $A$ is regular open set and every regular open set is $\alpha$–open set
Therefore $A$ is $m_{1-\alpha-open}$ set

Example 3.12:
Let $X=\{a,b,c,d\}$ and the topology be $T=\{\emptyset,\{a,d\},\{b,c\},\{d\},\{b,c,d\},X\}$
$RO(X)=\{\emptyset,\{a,d\},\{b,c\},\{d\},\{b,c,d\},X\}$
$AO(X)=\{\emptyset,\{a,d\},\{b,c\},\{d\},\{b,c,d\},X\}$
$A=\{b,c\}$ is $m_{1-pr-open}$ set but not $m_{1-\alpha-open}$ set

Proposition 3.13:
Every minimal open set is minimal pre regular open set. but the converse is not true.

Proof:
Let $A$ is a minimal open set. To prove $A$ is a minimal pre regular open set.
Since $A$ is open set and every open set is pre regular open set
Then minimal open set is minimal pre regular open set
Therefore $A$ is a minimal pre regular open set
Example 3.14:
Let \( X = \{a, b, c, d\} \) and the topology be \( T = \{\emptyset, [b,c],[a], [a,b,c],X\} \)
\( PRo(X) = \{\emptyset, [b,c],[a], [a,b,c],X\} \)
\( A = \{b\} \) is \( m_i-\overline{r-open} \) set but not \( m_i-\bar{o}pen \) set

Proposition 3.15:
Every minimal regular open set is minimal open set. but the converse is not true.

Proof:
Let \( A \) is a minimal regular open set. To prove \( A \) is a minimal open set.
Since \( A \) is regular open set and every regular open set is open set
Then minimal regular open set is minimal open set
Therefore \( A \) is minimal open set \( \Box \)

Example 3.16:
Let \( X = \{a, b, c, d\} \) and the topology be \( T = \{\emptyset, [a,c,d],[b],[a,b],X\} \)
\( Ro(X) = \{\emptyset, [a,c,d],[b],X\} \)
\( A = \{b\} \) is \( m_i-\overline{R-open} \) set but not \( m_i-\bar{R-open} \) set

Now we introduce the diagram which is explain the relation among these concepts and prove it as a propositions

Diagram 3.1.A

Definition 3.17:
A subset \( A \) of a space \( X \) is called to be generalized regular minimal closed [resp. generalized regular maximal open] set (briefly, \( g^r m_i-\overline{c}(X) \) , \( g^r m_i-\overline{o}(X) \) ) if \( U \subseteq Cl (A) \) [resp. \( Int(A) \subseteq U \) ] whenever \( A \subseteq U \) [resp. \( U \subseteq A \) ] and \( U \) is minimal regular open [resp. maximal regular closed] subset of \( X \).

Example 3.18:
Let \( X = \{a, b, c, d\} \) and the topology be \( T = \{\emptyset, [b,c],[a], [a,b,c],X\} \)
\( RCl(b) = [b,c,d] \subseteq RCl(A) = A \) is \( g^r m_i-\overline{c}losed \) set
\( A^c = [a,c,d] \) , \( U^c = \{a,d\} \)
\( RInt(A^c) = \{a\} \subseteq U^c \)

Definition 3.19:
A subset \( A \) of a space \( X \) is called to be regular generalized minimal closed [resp. regular generalized maximal open] set (briefly, \( r g^r m_i-\overline{c}(X) \) , \( r g^r m_i-\overline{o}(X) \) ) if \( U \subseteq Cl (A) \) [resp. \( Int(A) \subseteq U \) ] whenever \( A \subseteq U \) [resp. \( U \subseteq A \) ] and \( U \) is minimal regular open [resp. maximal regular closed] subset of \( X \).

Recall Example 3.18

Definition 3.21:
A subset \( A \) of a space \( X \) is called to be a regular generalized minimal closed [resp. regular generalized maximal open] set (briefly, \( r g^r m_i-\overline{c}(X) \) , \( r g^r m_i-\overline{o}(X) \) ) if \( U \subseteq Cl (A) \) [resp. \( Int(A) \subseteq U \) ] whenever \( A \subseteq U \) [resp. \( U \subseteq A \) ] and \( U \) is minimal pre regular open [resp. maximal pre regular closed] subset of \( X \).
Recall Example 3.2

**Example 3.22:**
*PRo(X)={Ø, {a}, {b}, {a, b}, X}, A={a}, U={a}*

Cl(A)={a, c} ⇒ U⊆Cl(A) ⇒ A is g r m1-closed set

A is g r m1-open set because A = {b, c}, U = {b, c}, Int{b, c}={b} ⇒ Int(A) ⊆ U c

**Definition 3.23:**
A subset A of a space X is called to be a generalized r regular minimal closed [resp. generalized r regular maximal open] set (briefly, g r m1-c(X), g r m1-a(O(X))) if U⊆RCI(A) [resp. RInt(A) ⊆ U] whenever A⊆U [resp. U⊆A] and U is minimal pre regular open [resp. maximal pre regular closed] subset of X.

**Example 3.24:**
Let X = {a, b, c} and the topology be T = {Ø, {a}, {b, c}, X}

PRo(X)={Ø, {a}, {b, c}, X}, A={b}, U={b}

RCI(b)={b, c} ⇒ U⊆RCI(A) ⇒ A is g r m1-closed set

A is g r m1-open set because A = {a} and U = {a}

RInt{a}={a} ⇒ RInt(A) ⊆ U c

Now we introduce the explain the relation among them

**Proposition 3.25:**
Every generalized regular minimal closed set is regular generalized minimal closed set but the converse is not true.

**Proof:**
Let A is g r m1-c(X). To prove A is g r m1-c(X).

Since A is g r m1-c(X) ⇒ U⊆RCI(A)

Since RCI(A) ⊆ Cl(A) ⇒ U⊆RCI(A) ⇒ Cl(A) ⇒ U⊆Cl(A)

Then A is g r m1-c(X) ■

**Example 3.26:**
Let X = {a, b, c, d} and the topology be T = {Ø, {a}, {b, c}, {b, d}, {b}, X}

RCl{b}={b, c} ⇒ U⊆RCI(b) ⇒ A is g r m1-c(X) set but A is not g r m1-c(X) set because

RCl(b)={b, c, d} ⇒ RCl(b)≠{b} ⇒ A is not g r m1-open set.

**Proposition 3.27:**
Every regular generalized minimal closed set is generalized minimal closed set but the converse is not true.

**Proof:**
Let A is g r m1-c(X). To prove A is g r m1-c(X).

Since A is g r m1-c(X) ⇒ U⊆Cl(A)

Since every minimal regular open set is minimal open set

Then U is minimal open set

Therefore A is g r m1-c(X) ■

**Example 3.28:**
Let X = {a, b, c, d} and the topology be T = {Ø, {d}, {a, b, c}, {a, b, d}, X}, RCl{a}={Ø, {a, b, c}, {d}, X}, A={a}, U={a, b}, Cl(a)={a, b, c}, U⊆Cl(A) ⇒ A is g r m1-closed set but A is not g r m1-closed set because,

Cl(a)={a, b, c}, U⊆Cl(A) but U is not m1-r-open set.

**Proposition 3.29:**
Every generalized minimal closed set is regular generalized minimal closed set but the converse is not true.

**Proof:**
Let A is g r m1-c(X). To prove A is g r m1-c(X).
Since \( A \) is \( g^* m_{1-c}(X) \) \( \Rightarrow U \subseteq Cl(A) \)
Since every minimal open set is minimal prec regular open set
Therefore \( A \) is \( g^{**} m_{1-c}(X) \)

**Example 3.30:**
Let \( X = \{a, b, c\} \) and the topology be \( T = \{\emptyset, \{a\}, \{a, b\}, X\}, \) \( PRo(X) = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}, \) \( A = \{c\}\) \( U = \{c\}, Cl(c) = \{c\} \Rightarrow U \subseteq Cl(A) \Rightarrow A \) is \( g^{**} m_{1-c}(X) \) but \( A \) is not \( g^* m_{1-c}(X) \) because \( Cl(c) = \{c\} \Rightarrow U \subseteq Cl(A) \) but \( U \) is not \( m_{1-open} \)

**Proposition 3.31:**
Every generalized \( ** \) regular minimal closed set is regular generalized \( * \) minimal closed set but the converse is not true.

**Proof:**
Let \( A \) is \( g^{**} r-m_{1-c}(X) \). To prove \( A \) is \( g^{**} m_{1-c}(X) \) since \( A \) is \( g^{**} r-m_{1-c}(X) \) \( \Rightarrow U \subseteq Cl(A) \)
Since \( RCl(A) \subseteq Cl(A) \Rightarrow U \subseteq RCl(A) \subseteq Cl(A) \Rightarrow U \subseteq Cl(A) \) Therefore \( A \) is \( g^{**} m_{1-c}(X) \)
Recall Example 3.30

**Example 3.32:**
Let \( X = \{a, b, c\} \) and the topology be \( T = \{\emptyset, \{a\}, \{a, b\}, X\}, A = \{a\}, U = \{a\} \)
\( Cl(a) = X \Rightarrow U \subseteq Cl(A) \Rightarrow A \) is \( g^{**} m_{1-c}(X) \) but \( A \) is not \( g^* r-m_{1-c}(X) \) because \( RCl(a) = X \Rightarrow RCl(A) \neq (A) \Rightarrow A \) is not \( g^{**} r-m_{1-c}(X) \)

**Proposition 3.33:**
The relations between regular generalized \( * \) minimal closed set with generalized \( ** \) regular minimal closed set are independent
Recall Example 3.32

**Example 3.34:**
\( m_{1-Ro(X)} = \{\{a\}, \{b, c\}\} \), \( m_{1-PRo(X)} = \{\{a\}, \{b\}, \{c\}\} \), \( A = \{b, c\} \), \( U = \{b, c\} \)
\( Cl(b) = \{b, c\} \Rightarrow U \subseteq Cl(A) \Rightarrow U \) is \( m_{1-open} \) set \( \Rightarrow A \) is \( g^{**} m_{1-closed} \) set but \( A \) is not \( g^{**} r-m_{1-closed} \) set because \( RCl(b) = \{b, c\} \Rightarrow U \subseteq RCl(A) \) but \( U \) is not \( m_{1-pr-open} \) set \( \Rightarrow A \) is not \( g^{**} r-m_{1-closed} \) set
\( B = \{c\} \), \( U = \{c\} \), \( RC\{c\} = \{b, c\} \Rightarrow U \subseteq RCl(B) \) and \( U \) is \( m_{1-pr-open} \) set \( A \) is \( g^{**} r-m_{1-closed} \) set but \( B \) is not \( \Rightarrow g^{**} m_{1-closed} \) because \( Cl(c) = \{b, c\} \Rightarrow U \subseteq Cl(B) \) but \( U \) is not \( m_{1-pr-open} \) set \( \Rightarrow B \) is not \( g^{**} - m_{1-closed} \) set.

**Proposition 3.35:**
The relations between generalized \( * \) minimal closed set with generalized \( ** \) regular minimal closed set are independent
Recall Example 3.18

**Example 3.36:**
\( A = \{b\} \), \( U = \{b, c\} \), \( Cl(b) = \{b, c, d\} \Rightarrow U \subseteq Cl(A) \) \( \Rightarrow U \subseteq Cl(A) \) but \( U \) is \( m_{1-open} \) set \( \Rightarrow A \) is \( g^{**} m_{1-closed} \) set but \( A \) is not \( g^{**} r-m_{1-closed} \) set because \( RCl(b) = \{b, c, d\} \Rightarrow U \subseteq RCl(A) \) but \( U \) is not \( m_{1-pr-open} \) set \( \Rightarrow A \) is not \( g^{**} r-m_{1-closed} \) set
\( B = \{c\} \), \( U = \{c\} \), \( RC\{c\} = \{b, c, d\} \Rightarrow U \subseteq RCl(B) \) and \( U \) is \( m_{1-pr-open} \) set \( \Rightarrow B \) is \( g^{**} r-m_{1-closed} \) set but \( B \) is not \( g^{**} m_{1-closed} \) set because \( Cl(c) = \{b, c, d\} \Rightarrow U \subseteq Cl(B) \) but \( U \) is not \( m_{1-open} \) set \( \Rightarrow B \) is not \( g^{**} - m_{1-closed} \) set.

Now we introduce the diagram which is explain the relation among these concepts and prove it as a propositions

**Diagram 3.B**

\[ g^{**} m_{1-c}(X) \quad \text{and} \quad g^{**} r-m_{1-c}(X) \]

\[ g^{**} m_{1-c}(X) \quad \text{and} \quad g^{**} r-m_{1-c}(X) \]
**Theorem 3.37:**
A subset $A$ of $X$ is a regular generalized minimal closed set if and only if there exists an minimal regular open set $U$ containing $A$ such that $\text{Cl}(A) = \text{Cl}(U)$

**Proof:**
Let $A$ be a $rg^{-m_1-c}(X)$. To prove $\text{Cl}(A) = \text{Cl}(U)$.

From definition, $U \subseteq U$, $U$ is minimal regular open set $\Rightarrow U \subseteq \text{Cl}(A)$

Then $\text{Cl}(U) \subseteq \text{Cl}(A)$ from (1) $\Rightarrow \text{Cl}(U) \subseteq \text{Cl}(A)$……… (1)

But, $A \subseteq U$ $\Rightarrow \text{Cl}(A) \subseteq \text{Cl}(U)$……… (2)

From (1) and (2) we have $\text{Cl}(A) = \text{Cl}(U)$

**Conversely:**
Since $A \subseteq U$, $\text{Cl}(A) = \text{Cl}(U) \supseteq U$

Then $A$ is $rg^{-m_1-c}(X)$

**Remark 3.38:**
$\emptyset$ and $X$ are not regular generalized minimal closed set resp. regular generalized minimal closed set.

**Proof:**
$\emptyset \not\subseteq \text{any } m_{1-RO}(X)$ [resp. $m_{1-RO}(X), m_{1-PRO}(X), m_{1-RO}(X)$]

Similarly, $X \not\subseteq \text{any } m_{1-RO}(X)$ [resp. $m_{1-RO}(X), m_{1-PRO}(X), m_{1-RO}(X)$]

**Theorem 3.39:**
An arbitrary union of regular generalized minimal closed set resp. regular generalized minimal closed set is regular generalized minimal closed set if it is contained in a minimal regular open set.

**Proof:**
Let $\{A_i\}_{i \in I}$ be a collection of $rg^{-m_1-c}(X)$ [resp. $rg^{-m_1-c}(X), g^{-m_1-c}(X), g^{-m_1-c}(X)$] set

Let $\bigcup_i A_i \subseteq U$, $U$ is $m_{1-RO}(X)$ [resp. $m_{1-RO}(X), m_{1-RO}(X), m_{1-RO}(X)$]

Then $A_i \subseteq U$ for all $i \in I$

Since $\{A_i\}_{i \in I}$ are a collection of $rg^{-m_1-c}(X)$ [resp. $rg^{-m_1-c}(X), g^{-m_1-c}(X), g^{-m_1-c}(X)$] set

Then $\bigcup_i \text{Cl}(A_i) \subseteq \text{Cl}(A_i) \subseteq U$ [resp. $\bigcup_i \text{Cl}(A_i) \subseteq \text{Cl}(A_i) \subseteq U$]

So $\text{Cl}(\bigcup_i A_i) \supseteq \text{Cl}(A_i) \supseteq U$ [resp. $\text{Cl}(\bigcup_i A_i) \supseteq \text{Cl}(A_i) \supseteq U$]

So arbitrary union of $rg^{-m_1-c}(X)$ [resp. $rg^{-m_1-c}(X), g^{-m_1-c}(X), g^{-m_1-c}(X)$] set is $rg^{-m_1-c}(X)$ [resp. $rg^{-m_1-c}(X), g^{-m_1-c}(X), g^{-m_1-c}(X)$] set

**Remark 3.40:**
Finite intersection of regular generalized minimal closed set resp. regular generalized minimal closed set need not be regular generalized minimal closed set.

Recall Example 3.1.23

**Example 3.41:**
Let $X = \{a, b, c\}$

$T = \emptyset, \{a\}, \{b, c\}, X$

$T^* = \{X, \{b, c\}, \{a\}, \emptyset\}$

$\text{Ro}(X) = \emptyset, \{a\}, \{b, c\}, X$

$\text{Re}(x) = \{X, \{b, c\}, \{a\}, \emptyset\}$

$A = \{b, c\} \Rightarrow A$ is $rg^{-m_1-c}(closed)$ set [resp. $g^{-r-m_1-c}(closed)$ set] of $X$, but

$B = \{a\} \Rightarrow B$ is $rg^{-m_1-c}(closed)$ set [resp. $g^{-r-m_1-c}(closed)$ set] of $X$, but

$A \cap B = \emptyset$ is not $rg^{-m_1-c}(closed)$ set [resp. $g^{-r-m_1-c}(closed)$ set] of $X$. 

...
Theorem 3.42:
Non-null intersection of regular generalized minimal closed [resp. generalized regular minimal closed]
set and closed set is regular generalized minimal closed [resp. generalized regular minimal closed] set.

Proof:
Let \( A \) be \( r*g^*m_{1-c(X)} \) [resp. \( g^*r^{-}m_{1-c(X)} \)] set and let \( F \) be a closed set \( U \) is \( m_{1-RG(X)} \) Since \( A \) is \( r*g^*-m_{1-c(X)} \) [resp. \( g^*r^{-}m_{1-c(X)} \)] set
Therefore \( U \subseteq Cl(A) \) [resp. \( U \subseteq RCl(A) \)]. \( \Rightarrow U \subseteq Cl(A) \cap F \) [resp. \( U \subseteq RCl(A) \cap F \)]
\( \Rightarrow U \subseteq Cl(A \cap F) \) [resp. \( U \subseteq RCl(A \cap F) \)].
Then \((A \cap F)\) is \( r*g^*m_{1-c(X)} \) [resp. \( g^*r^{-}m_{1-c(X)} \)] set.

Theorem 3.43:
Let \( A \) be a regular generalized minimal closed set and \( B \) be a subset of \( X \) contained in the same
minimal regular open set
If \( A \subseteq B \subseteq Cl(A) \), then \( B \) is also regular generalized minimal closed set.

Proof:
Let \( A \) be a \( r*g^*m_{1-c(X)} \) then \( A \subseteq U, U \subseteq Cl(A) \)
Then \( Cl(U) \subseteq Cl Cl(A) \Rightarrow Cl(U) \subseteq Cl(A) \) \( \cdots \cdots \cdots \cdots (1) \)
But, \( A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U) \cdots \cdots \cdots \cdots (2) \)
From (1) and (2) we have
\( Cl(A) = Cl(U) \)
\( B \) is subset of \( X \) \( \Rightarrow B \subseteq U \), here \( A \subseteq B \subseteq Cl(A) \)
i.e \( Cl(A) = Cl(B) = Cl(U) \)
Therefore \( B \) also a \( r*g^{-}m_{1-c(X)} \) ■

Theorem 3.44:
A proper non empty subset \( F \) of \( X \) is regular generalized minimal closed set iff \((X-F)\) is regular
generalized maximal open set.

Proof:
Let \( F \) be a \( r*g^{-}m_{1-c(X)} \).To prove \((X-F)\) is a \( r*g^*-m_{a-o(X)} \).
Suppose \((X-F)\) is not is \( r*g^{-}m_{a-o(X)} \).
Then \( \exists \) a minimal regular open set \( U \subseteq (X-F) \Rightarrow (X-F) \subseteq U \)
Therefore \((X-U) \subseteq F\), and \((X-U)\) is a maximal regular closed set which is a contradiction for \( F \) is a \( r*g^*-m_{1-c(X)} \) ■

Conversely:
Let \((X-F)\) be a \( r*g^*-m_{a-o(X)} \).To prove \( F \) is a \( r*g^*m_{1-c(X)} \).
Suppose \( F \) is not \( r*g^*m_{1-c(X)} \).Then \( \exists \) a maximal regular closed set \( E \neq F \) Such that \( X \neq E \subseteq F \).That is \((X-F) \subseteq (X-E)\), and \((X-E)\) is a minimal regular open set which is a contradiction for \((X-F)\) a \( r*g^*m_{a-o(X)} \).
Therefore \( F \) is \( r*g^*m_{1-c(X)} \) ■.

Theorem 3.45:
A subset \( A \) of \( X \) is a generalized regular minimal closed set iff \( \exists \) a minimal regular open set \( U \)
containing \( A \) such that \( RCl(A) = Cl(U) \)

Proof:
Let \( A \) be a \( g^*r^{-}m_{1-c(X)} \).To prove \( RCl(A) = Cl(U) \).
From definition \( A \subseteq U, U \) is minimal regular open set \( \Rightarrow U \subseteq RCl(A) \)
Then \( Cl(U) \subseteq Cl [RCl(A)] \Rightarrow Cl(U) \subseteq RCl(A) \) \( \cdots \cdots \cdots \cdots (1) \)
But, \( A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U) \)
Since \( RCl(A) \subseteq Cl(A) \).
Then \( RCl(A) \subseteq Cl(A) \subseteq Cl(U) \Rightarrow RCl(A) \subseteq Cl(U) \) \( \cdots \cdots \cdots \cdots (2) \)
From (1) and (2) we have
\( RCl(A) = Cl(U) \)
Conversely:
Since $A \subseteq U, RCl(A) = Cl(U) \supseteq U$
Therefore $A$ is $g \ast r_m_{1-c(X)}$

**Theorem 3.46:**
Let $A$ be a generalized regular minimal closed set and $B$ be a subset of $X$ contained in the same minimal regular open set, if $A \subseteq B \subseteq RCl(A)$, then $B$ is also generalized regular minimal closed set.

**Proof:**
Let $A$ be a $g \ast r_m_{1-c(X)}$ then $A \subseteq U, U \subseteq RCl(A)$
Then $Cl(U) \subseteq Cl(RCl(A)) \Rightarrow Cl(U) \subseteq RCl(A)$ \quad \cdots \cdots (1)
But, $A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U)$
Since $RCl(A) \subseteq Cl(A)$
Then $RCl(A) \subseteq Cl(A) \subseteq Cl(U) \Rightarrow RCl(A) \subseteq Cl(U)$ \quad \cdots \cdots (2)
From (1) and (2) we have
$RCl(A) = Cl(U)$.
Since $B$ is subset of $X \Rightarrow B \subseteq U \Rightarrow Cl(B) \subseteq Cl(U)$
Since $RCl(B) \subseteq Cl(Cl(B)) \subseteq Cl(U)$
But $A \subseteq B \Rightarrow RCl(A) \subseteq Cl(B) \subseteq RCl(A)$
Since $RCl(A) \subseteq Cl(A) \subseteq RCl(B) \subseteq Cl(B) \subseteq RCl(A) = Cl(U)$
$Cl(U) \subseteq RCl(B) \subseteq Cl(U)$
\i.e $Cl(U) = RCl(B) = RCl(A)$
Then $U \subseteq Cl(U) = RCl(B) = RCl(A)$
Therefore $B$ a $g \ast r_m_{1-c(X)}$

**Theorem 3.47:**
A proper non empty subset $F$ of $X$ is generalized regular minimal closed set iff $(X-F)$ is generalized regular maximal open set.

**Proof:**
Let $F$ be a $g \ast r_m_{1-c(X)}$. To prove $(X-F)$ is a $g \ast r_m_{a-o(X)}$.
Suppose $(X-F)$ is not is a $g \ast r_m_{a-o(X)}$
Then $\exists$ a minimal regular open set $U \neq (X-F) \Rightarrow (X-F) \subseteq U$
Then $(X-U) \subseteq F$, and $(X-U)$ is a maximal regular closed set which is a contradiction for $F$ is a $g \ast r_m_{1-c(X)}$

Conversely:
Let $(X-F)$ be a $g \ast r_m_{a-o(X)}$. To prove $F$ is a $g \ast r_m_{1-c(X)}$
Suppose $F$ is not $g \ast r_m_{1-c(X)}$. Then $\exists$ a maximal regular closed set $E \neq F$ Such that $X \neq E \subseteq F$. That is $(X-F) \subseteq (X-E)$, and $(X-E)$ is a minimal regular open set which is a contradiction for $(X-F)$ a $g \ast r_m_{a-o(X)}$
Therefore $F$ is $g \ast r_m_{1-c(X)}$

**Theorem 3.48:**
A subset $A$ of $X$ is a regular generalized minimal closed set iff $\exists$ a minimal pre regular open set $U$ containing $A$ such that $Cl(A) = Cl(U)$.

**Proof:**
Let $A$ be a $rg \ast r_m_{1-c(X)}$. To prove $Cl(A) = Cl(U)$.
From definition $A \subseteq U, U$ is minimal pre regular open set $\Rightarrow U \subseteq Cl(A)$
Then $Cl(U) \subseteq Cl(Cl(A)) \Rightarrow Cl(U) \subseteq Cl(A)$ \quad \cdots \cdots (1)
But, $A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U)$ \quad \cdots \cdots (2)
From (1) and (2) we have
$Cl(A) = Cl(U)$

Conversely:
Since $A \subseteq U, Cl(A) = Cl(U) \supseteq U$
Therefore $A$ is $rg \ast r_m_{1-c(X)}$
Remark 3.49:
Finite intersection of regular generalized minimal closed set need not be regular generalized minimal closed set which follows from the following example
Recall Example 3.1.23

Example 3.50:
\[ m_{1-pro}(X) = \{a\}, \{b\}, \{c\} \]
\[ m_{a-prc}(X) = \{b,c\}, \{a,c\}, \{a,b\} \]
\[ A = \{a\} \] is rg \( m_{l-}\)-closed \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set of \( X \)
\[ B = \{c\} \] is rg \( m_{l-}\)-closed \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set of \( X \), but
\[ A \cap B = \emptyset \] is not rg \( m_{l-}\)-closed \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set of \( X \).

Theorem 3.51:
Non-null intersection of regular generalized minimal closed set and closed set is regular generalized minimal closed set.

Proof:
Let \( A \) be \( g^{**} \text{rg } m_{l-}\text{-closed} \) \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set and let \( F \) be a closed set. \( U \) is \( m_{l-\text{pro}}(X) \).
Since \( A \) is \( g^{**} \text{rg } m_{l-}\text{-closed} \) \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set
Then \( U \subseteq \text{Cl}(A) \) \[ \text{resp. } U \subseteq \text{Cl}(A) \]
\[ \Rightarrow U \subseteq \text{Cl}(A) \cap F \] \[ \text{resp. } U \subseteq \text{Cl}(A) \cap F \],
Therefore \( (A \cap F) \) is \( g^{**} \text{rg } m_{l-}\text{-closed} \) \[ \text{resp. } g^{**} \text{rg } m_{l-}\text{-closed} \] set.

Theorem 3.52:
Let \( A \) be a regular generalized minimal closed set and \( B \) be a subset of \( X \) contained in the same minimal pre regular open set If \( A \subseteq B \subseteq \text{Cl}(A) \), then \( B \) is also regular generalized minimal closed set.

Proof:
Let \( A \) be a \( g^{**} \text{rg } m_{l-}\text{-closed} \) then \( A \subseteq U \), \( U \subseteq \text{Cl}(A) \)
But \( A \subseteq U \Rightarrow \text{Cl}(A) \subseteq \text{Cl}(U) \) \[ \text{(1)} \]
From (1) and (2) we have \( \text{Cl}(A) = \text{Cl}(U) \)
\( B \) is subset of \( X \) \[ \Rightarrow B \subseteq U \] , here \( A \subseteq B \subseteq \text{Cl}(A) \)
i.e. \( \text{Cl}(B) = \text{Cl}(U) \)
Therefore \( B \) also a \( g^{**} \text{rg } m_{l-}\text{-closed} \).

Theorem 3.53:
A proper non-empty subset \( F \) of \( X \) is regular generalized minimal closed set \( \text{iff } (X-F) \) is regular generalized maximal open set.

Proof:
Let \( F \) be a \( g^{**} \text{rg } m_{l-}\text{-closed} \). To prove \( (X-F) \) is \( g^{**} \text{rg } m_{l-}\text{-closed} \).
Suppose \( (X-F) \) is not a \( g^{**} \text{rg } m_{l-}\text{-closed} \).
Then \( (X-F) \) is a minimal regular open set \( U \cap (X-F) \subseteq U \)
Then \( (X-U) \subseteq F \), and \( (X-U) \) is a maximal pre regular closed set which is a contradiction for \( F \) is a \( g^{**} \text{rg } m_{l-}\text{-closed} \).

Conversely:
Let \( (X-F) \) be a \( g^{**} \text{rg } m_{l-}\text{-closed} \). To prove \( F \) is a \( g^{**} \text{rg } m_{l-}\text{-closed} \).
Suppose \( F \) is not \( g^{**} \text{rg } m_{l-}\text{-closed} \). Then \( F \) is a maximal pre regular closed set \( E \neq F \) such that \( E \subseteq F \).
That is \( (X-F) \subseteq (X-E) \), and \( (X-E) \) is a minimal pre regular open set which is a contradiction for \( (X-F) \) a \( g^{**} \text{rg } m_{l-}\text{-closed} \). Therefore \( F \) is a \( g^{**} \text{rg } m_{l-}\text{-closed} \).
**Theorem 3.54:**
A subset $A$ of $X$ is a generalized regular minimal closed set iff $\exists$ a minimal pre regular open set $U$ containing $A$ such that $RCl(A)=Cl(U)$

**Proof:**
Let $A$ be a $g^{**}r_m_{i-c}(X)$. To prove $RCl(A)=Cl(U)$.
From definition $A \subseteq U \subseteq RCl(A)$
Then $Cl(U) \subseteq Cl[\{RCl(A)\}] \Rightarrow Cl(U) \subseteq RCl(A)$ \hspace{1cm} (1)
But, $A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U)$
Since $RCl(A) \subseteq Cl(A)$
Then $RCl(A) \subseteq Cl(A) \subseteq Cl(U) \Rightarrow RCl(A) \subseteq Cl(U)$ \hspace{1cm} (2)
From (1) and (2) we have $RCl(A)=Cl(U)$

Conversely:
Since $A \subseteq U \subseteq RCl(A)$ $\Rightarrow U \supseteq U$
Therefore $A$ is $g^{**}r_m_{i-c}(X)$

**Theorem 3.55:**
Let $A$ be a generalized regular minimal closed set and $B$ be a subset of $X$ contained in the same minimal pre regular open set.
If $A \subseteq B \subseteq RCl(A)$, then $B$ is also generalized regular minimal closed set.

**Proof:**
Let $A$ be a $g^{**}r_m_{i-c}(X)$ then $A \subseteq U \subseteq RCl(A)$
Then $Cl(U) \subseteq Cl[\{RCl(A)\}] \Rightarrow Cl(U) \subseteq RCl(A)$ \hspace{1cm} (1)
But, $A \subseteq U \Rightarrow Cl(A) \subseteq Cl(U)$
Since $RCl(A) \subseteq Cl(A)$ Then $RCl(A) \subseteq Cl(A) \subseteq Cl(U) \Rightarrow RCl(A) \subseteq Cl(U)$ \hspace{1cm} (2)
From (1) and (2) we have $RCl(A)=Cl(U)$.

If $B$ is subset of $X \Rightarrow B \subseteq U \Rightarrow Cl(B) \subseteq Cl(U)$
Since $RCl(B) \subseteq Cl(B) \subseteq Cl(U) \Rightarrow RCl(B) \subseteq Cl(U)$,
But $A \subseteq B \subseteq RCl(A) \Rightarrow Cl(A) \subseteq Cl(B) \subseteq RCl(A)$
Since $RCl(A) \subseteq Cl(A) \subseteq RCl(B) \subseteq Cl(B) \subseteq RCl(A) \Rightarrow Cl(U)$
$Cl(U) \subseteq RCl(B) \subseteq Cl(U)$
$i.e Cl(U)=RCl(B)=RCl(A)$
Then $U \subseteq Cl(U) \Rightarrow RCl(B)=RCl(A)$

**Theorem 3.56:**
A proper non empty subset $F$ of $X$ is generalized regular minimal closed set iff $(X-F)$ is generalized regular maximal open set.

**Proof:**
Let $F$ be a $g^{**}r_m_{i-c}(X)$. To prove $(X-F)$ is a $g^{**}r_m_{a-o}(X)$.
Suppose $(X-F)$ is not a $g^{**}r_m_{a-o}(X)$
Then $\exists$ a minimal pre regular open set $U \neq (X-F) \Rightarrow (X-F) \subseteq U$
Then $(X-U) \subseteq F$, and $(X-U)$ is a maximal pre regular closed set which is a contradiction for $F$ is a $g^{**}r_m_{i-c}(X)$.

Conversely:
Let $(X-F)$ be a $g^{**}r_m_{a-o}(X)$. To prove $F$ is a $g^{**}r_m_{i-c}(X)$.
Suppose $F$ is not $g^{**}r_m_{i-c}(X)$. Then $\exists$ a maximal pre regular closed set $E \neq F$ Such that $X \neq E \subseteq F$.
That is $(X-F) \subseteq (X-E)$, and $(X-E)$ is a minimal pre regular open set which is a contradiction for $(X-F)$ a $g^{**}r_m_{a-o}(X)$. Therefore $F$ is $g^{**}r_m_{i-c}(X)$.

**REFERENCES**

Bhattacharya, S., Halder, S. 2011. Study on generalized regular open sets in ordinary topological space, India, Department of Mathematics, Tripura University, Suryamaninagar, Tripura – 99130.