Australian Journal of Basic and Applied Sciences, 8(11) Special 2014, Pages: 23-29

AENSI Journals

Australian Journal of Basic and Applied Sciences

ISSN:1991-8178

Journal home page: www.ajbasweb.com

Extrapolation of Boundary Value Problems

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ARTICLE INFO

Article history:
Received 25 June 2014
Received in revised form
8 July 2014
Accepted 10 August May 2014
Available online 30 August 2014

Keywords:
Extrapolation, Symmetric
Boundary Value Problem
Integrating factor

ABSTRACT

In this article, transformation involving the use of an integrating factor before applying the symmetric discretization is given for two point boundary value problems. The extrapolations of linear BVPs using an integrating factor are found to have much better accuracy than those applied without the integrating factor. Results of numerical experiments are presented which shows the effect of the integrating factor in improving the accuracy of extrapolation.

To Cite This Article: Annie Gorgey. Extrapolation of Boundary Value Problems. ***Aust. J. Basic & Appl. Sci.,*** *8(11): 23-29, 2014*

INTRODUCTION

Extrapolation has always been an important technique for accelerating the convergence of solutions arising from discretization methods. It has been successfully applied to numerical quadrature (Romberg, 1955) and the solution of ODEs. Applications of extrapolation to ordinary differential equations (ODEs) became popular when Gragg (1965) proved the existence of an asymptotic error expansion for the explicit midpoint rule. Gragg also introduced a smoothing formula to dampen the effects of the parasitic component in the numerical solution. This smoothing formula has been successfully used by Bulirsch and Stoer (1966) in the numerical solution of nonstiff problems. Following this success, Lindberg (1971) and Bader and Deuflhard (1983) extended the application of extrapolation to stiff problems.

On the other hand, applications of extrapolations applied to boundary value problems (BVPs) are given by Fox in 1957. He investigated the deferred approach for the various boundary value problems using second order central difference formula. Laurent (1963, 1964) then solved the boundary value problems (BVPs) in ODEs using iterative shooting methods and also PDE problem using central differences. Meanwhile Stetter (1965) extended Gragg's theory for solving integral equation, PDEs and BVPs in ODEs while Keller (1969) had used central difference method to solve the linear ODEs and parabolic PDEs and showed that extrapolation can obtained high order accuracy approximations. There are also recent papers on solving PDEs using extrapolation technique (Dai, 2014) and (Liu, 2013).

Although the idea of extrapolation is old, extrapolation technique has shown to be a powerful tool in improving the accuracy of numerical solutions especially when solving symmetric methods (Gorgey, 2012). This is because each successive extrapolation increases the order by two instead of one at a time. For example, an order-2 discretization will give order 4 for the first extrapolation and order 6 for the second extrapolation.

In this article, symmetric discretization for both the first and second derivative is given. However, instead of using a symmetric discretization for y' directly using the central difference method, a transformation involving the use of an integrating factor before applying the symmetric discretization is done.

Boundary Value Problems:

A two point boundary value problems (BVPs) has the form

$$y'' = f(x, y(x), y'(x)), \qquad 0 \le x \le 1, \tag{1}$$

$$y(0) = \alpha, \quad y(0) = \beta. \tag{2}$$

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Consider linear problems of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \qquad 0 \le x \le 1, \tag{3}$$

where p, q and f are functions of x. The boundary conditions are given in (2). Equation (3) is assumed to have a unique solution that is at least twice continuously differentiable.

One simple approach to solve equation (3) is numerically by employing finite difference scheme.

Solution By Finite Difference Scheme:

The numerical procedure using finite difference approach is done by dividing the interval [0,1] to *n+1* sub-intervals of equal length h where h is the also the grid spacing. The grid points are denoted by x_i while the boundary points are x_0 and x_{n+1}. Approximating the second derivative y'' by finite differences at the points x_i gives the order-2 discretization as follows:

$$y''(x_i) \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}, \tag{4}$$

For the first derivative y', it is approximated using the central difference approximation

$$y'(x_i) \approx \frac{y(x_{i+1}) - y(x_{i-1})}{2h}.$$

Now, substituting into equation (3) and denoting $q_i = q(x_i)$, $p_i = p(x_i)$ and $f_i = f(x_i)$, gives

$$\frac{1}{h^2}\left(y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right) + p_i\left(\frac{y(x_{i+1}) - y(x_{i-1})}{2h}\right) + q_i y(x_i) \approx f_i, \quad i = 1,2,\ldots,n. \tag{5}$$

Equation (5) can be written more precisely as

$$\left(1 - \frac{p_i h}{2}\right)y_{i-1} + (-2 + h^2 q_i)y_i + \left(1 + \frac{p_i h}{2}\right)y_{i+1} \approx h^2 f, \quad i = 1,2,\ldots,n. \tag{6}$$

Equation (6) then forms a system of n linear equations in the n unknowns $y_1,\ldots,y_n$. In a matrix form, it can be written as

$$\begin{bmatrix} -2+q_1h^2 & 1+\frac{p_1h}{2} & 0 & \cdots & 0 \\ 1-\frac{p_2h}{2} & -2+q_2h^2 & 1+\frac{p_2h}{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -2+q_{n-1}h^2 & 1+\frac{p_{n-1}h}{2} \\ 0 & \cdots & 0 & 1-\frac{p_nh}{2} & -2+q_nh^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} h^2 f_1 - \left(1-\frac{p_1h}{2}\right)\alpha \\ h^2 f_2 \\ \vdots \\ h^2 f_{n-1} \\ h_2 f_n - \left(1+\frac{p_nh}{2}\right)\beta \end{bmatrix}.$$

Now the above finite difference approach is not something new. However, instead of using finite difference scheme, equation (3) can also be solved using integrating factor technique which is discussed in the next topic.

Solution By Integrating Factor Technique:

Most mathematicians are familiar with the solution of the first order ODEs using the integrating factor technique. However, it is also possible to solve equation (3) using this technique. The idea is to discretize the first derivative until the equation does not contain any first derivative. Step by step transformations using integrating factor are given below:

Step 1:

Define an integrating factor $J(x) = e^{\int p(x)dx}$. Equation (3) becomes

$$\frac{d(J(x)y'(x))}{dx} + J(x)q(x)y(x) = J(x)f(x).$$

Step 2:

Discretize the derivative using the symmetric order-2 formula over a half step.

$$\frac{J_{i+1/2}y'_{i+1/2} - J_{i-1/2}y'_{i-1/2}}{h} + J_i q_i y_i = J_i f_i, \quad i = 1,2,\ldots,n.$$

Step 3:

Discretize the derivative y' again using the symmetric order-2 formula over a half step.

$$J_{i+1/2}(y_{i+1} - y_i) - J_{i-1/2}(y_i - y_{i-1}) + h^2 J_i q_i y_i = h^2 J_i f_i,$$
$$J_{i-1/2}y_{i-1} - (J_{i-1/2} + J_{i+1/2} - h^2 J_i q_i)y_i + J_{i+1/2}y_{i+1} = h^2 J_i f_i,$$

Step 4:

For simplicity, let $D_i = J_{i-1/2} + J_{i+1/2} - h^2 J_i q_i$. By doing so, the equation in Step 3 also give a tridiagonal system in the following matrix form.

$$\begin{bmatrix} -D_1 & J_{3/2} & 0 & \cdots & 0 \\ J_{3/2} & -D_2 & J_{5/2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & J_{n-3/2} & -D_{n-1} & J_{n-1/2} \\ 0 & \cdots & 0 & J_{n-1/2} & -D_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} -J_{1/2}\alpha + h^2 J_1 f_1 \\ h^2 J_2 f_2 \\ \vdots \\ h^2 J_{n-1} f_{n-1} \\ -J_{n+1/2}\beta + h_2 J_n f_n \end{bmatrix}.$$

The above procedure can also be carried out for higher order discretization. In this article, the numerical results are given for order-2 discretization with and without integrating factor. In addition to this approach, since the method is symmetric, extrapolation will take advantage. Symmetric in the sense of Stetter (1973) also means it has a global discretization error with an asymptotic expansion containing even powers of *h*. This special property gave extra benefits to extrapolation method because each successive extrapolation increases the order by 2 at a time. For detailed explanations on extrapolation of ODEs and symmetric methods, one may refer to (Chan & Gorgey, 2013) and (Gorgey, 2012). Hence, the numerical results are also given for extrapolated order-2 discretization with and without integrating factor. The next topic gives the numerical experiments for three types of boundary value problems are the following observations are made.

Numerical Results And Discussion:

The numerical results are given for two types of boundary value problems. In all cases, the exact solutions are known therefore the graphs of error versus the grid points are given. To check the accuracy of the extrapolation, the graphs of error versus the grid spacing *h* are also given.

Problem 1:

$$y'' + 3y' + (25/4)y = 0.$$
$$y(0) = 0, y(1) = 1.$$
$$y(x) = \exp(3/2(1-x))\sin(2x)/\sin(2).$$

Problem 1 shows that extrapolation gives better accuracy when is used for the integrating factor (IF) technique instead of the second order difference formula. The results are given in Figure 1, Figure 2 and Figure 3.

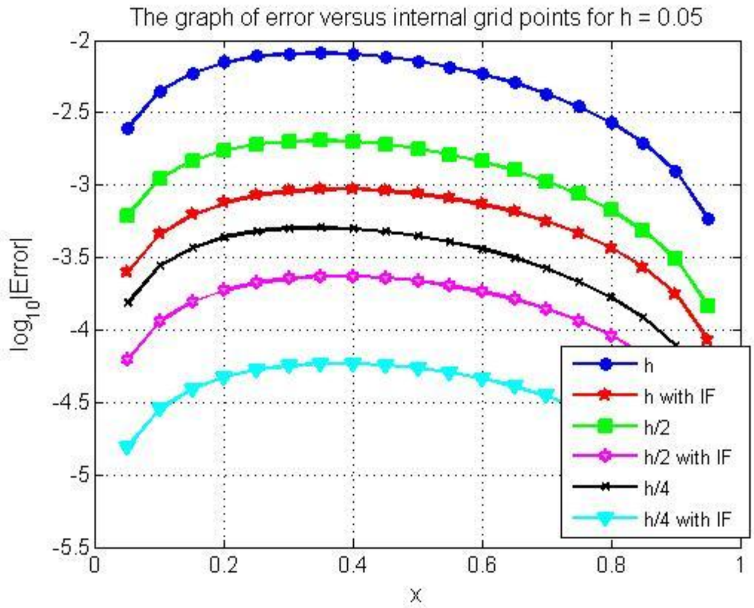

Fig. 1: Problem 1 with and without integrating factor technique.

Figure 1 shows the behavior diagram of Problem 1 with and without integrating factor. The result shows that the errors are smaller when solved using integrating factor instead of second order approximations.

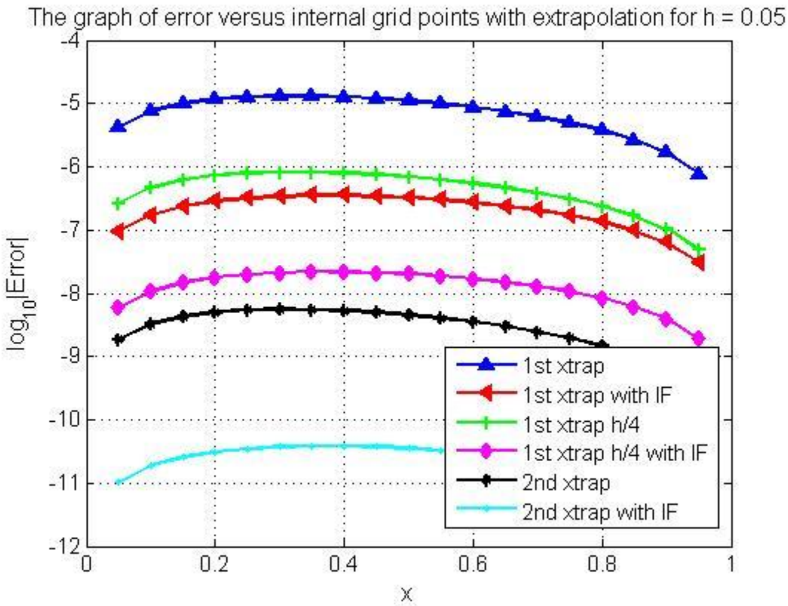

Fig. 2: Problem 1 with and without integrating factor/extrapolation technique.

Figure 2 shows the behavior diagram of Problem 1 with and without integrating factor and extrapolation. According to Figure 2, the errors are smaller when solving using integrating factor technique together with extrapolation if compared the solution without integrating factor and extrapolation.

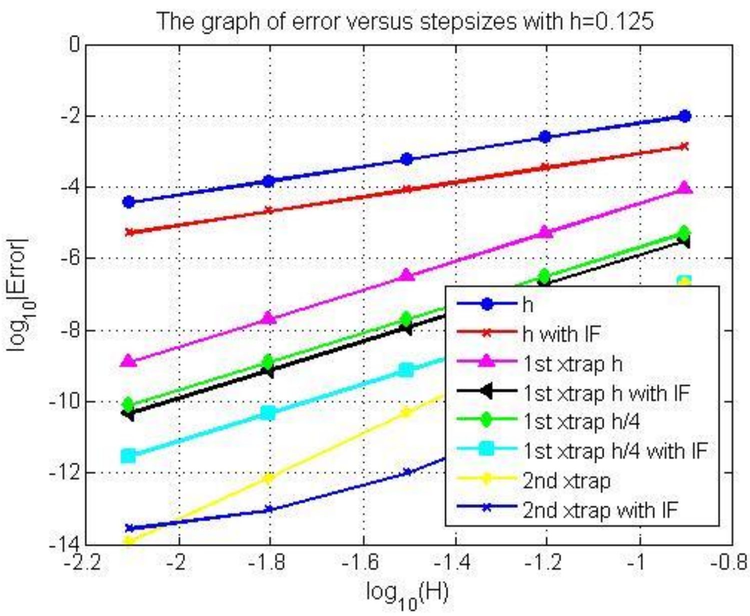

Fig. 3: Accuracy diagram of Problem 1with and without integrating factor/extrapolation technique.

Figure 3 shows the accuracy diagram of Problem 1 with and without integrating factor and extrapolation. In Figure 3, it is shown that by the slope of the graph, the order of the first level extrapolation is 4 while the order of the second level extrapolation is 6. The second level extrapolation with integrating factor is more accurate than the second level extrapolation without integrating factor.

Problem 2:

$$xy'' - y' - x = 0.$$

$$y(0) = 0, y(1) = 0.$$

$$y(x) = \frac{1}{2}x^2 \ln(x).$$

Problem 2 is given by (L, Fox, 1962). This is an interesting problem where according to Fox extrapolation will fail if used with central difference method however with integrating factor, extrapolation is shown to work efficiently.

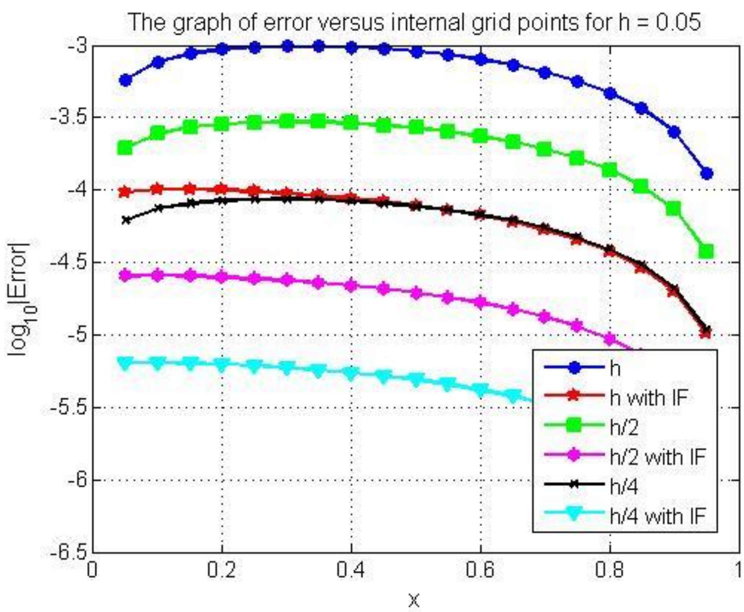

Fig. 4: Problem 2 with and without integrating factor technique.

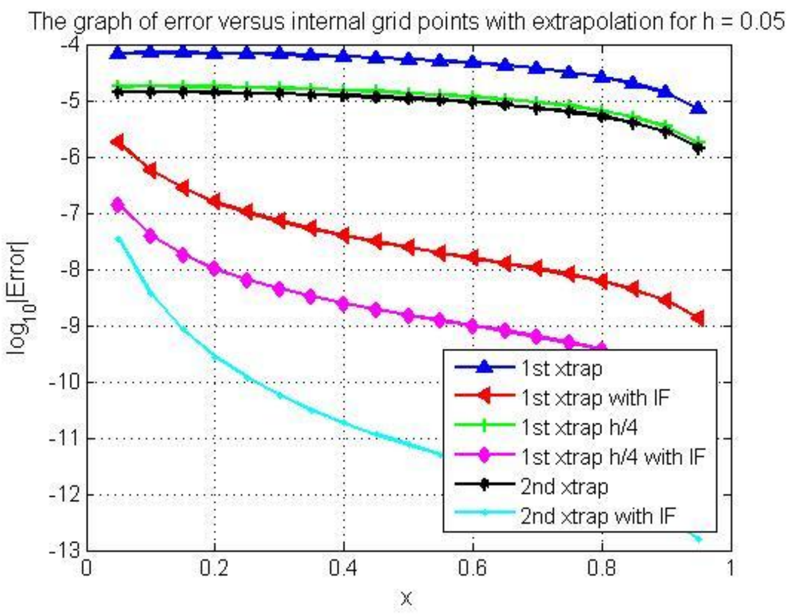

Fig. 5: Problem 2 with and without integrating factor/extrapolation technique.

Figure 4 and 5 show the behavior diagrams of Problem 2 with and without integrating factor and integrating factor with extrapolation. Figure 5 shows an interesting results where although extrapolation fails when solved using second order approximations which is true according to Fox, however, extrapolation works well when solved using integrating factor. This is indeed one new observation.

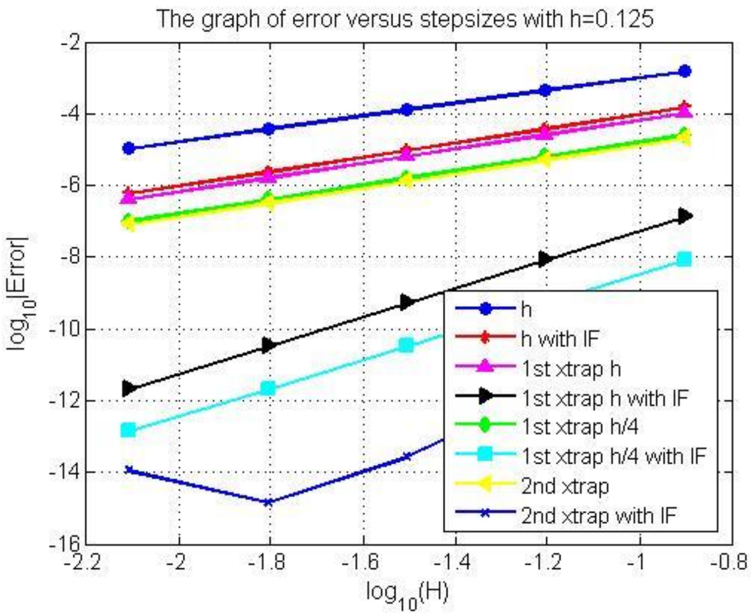

Fig. 6: Accuracy diagram of Problem 2 with and without integrating factor/extrapolation technique.

Similar observation is shown in Figure 6 where the solutions fail when solved with extrapolation using second order approximations. On the other hand, extrapolation works well when solved using integrating factor. The slope of the graph gives the first and second level extrapolations of the integrating technique are of order-4 and order-6 respectively.

Conclusions:

From the numerical experiments it is therefore shown that solving y' using IF technique have much better accuracy than those solved using the second order central difference formula. In addition to that, extrapolation with integrating factor is more efficient for BVP. Interestingly extrapolation fails for the problem

$xy'' - y' - x = 0$ when solved using the second order difference formula but works well for the IF technique. Therefore, it can be concluded that integrating factor technique with extrapolation preserve the asymptotic errors expansions of the solutions and therefore gives efficient results. It is therefore interesting to try on various BVPs as well as PDEs and see whether similar observations can be obtained.

ACKNOWLEDGEMENTS

This research was supported by the research grant RAGS, Vote No: 2013-0004-107-72. The author also would like to thank Dr. Robert Chan for the ideas and suggestions given while conducting this research.

REFERENCES

Bader, G. and P. Deuflhard, 1983. A semi-implicit mid-point rule for stiff systems of ordinary differential equations. Numer. Math., 41: 373-398.

Bulirsch, R. and J. Stoer, 1966. Numerical treatment of ordinary differential equations by extrapolation method. Numer. Math, 8: 1-13.

Chan, R.P.K. and A. Gorgey, 2013. Active and passive symmetrization of Runge-Kutta Gauss methods. Applied Numerical Mathematics, 67: 64-77.

Dai, R., 2014. *Richardson extrapolation-based high accuracy high efficiency computation for partial differential equations*, PhD Thesis. University of Kentucky.

Fox, L., 1957. *The Numerical Solution of Two-Point Boundary Value Problems in Ordinary Differential Equations*. Oxford University Press, London.

Fox, L., 1962. *Numerical Solutions of Ordinary and Partial Differential Equations, Based on Summer School*. Pergamon Press, Oxford, London.

Gorgey, A., 2013. *Extrapolation of symmetrized Runge-Kutta methods*, PhD Thesis, University of Auckland.

Gragg, W., 1965. On extrapolation algorithm for ordinary IVPs. SIAM J. Numer. Anal, 2: 384-403.

Keller, H.B., 1969. Accuracy difference methods for linear ordinary differential systems subject to linear constraints. SIAM J. Numer. Anal., 6: 8-30.

Laurent, P.J., 1963. Convergence du proédé d'extrapolation de Richardson. Troisième Congrès de 1'AFCALTI, Toulouse, 81-98.

Laurent, P.J., *Ětude des proédé d'extrapolation en analyse numérique*. Thesis, Univ. of Grenoble, Grenoble, France.

Lindberg, B., 1971. On the smoothing and extrapolation for the trapezoidal rule. BIT, 11: 29-52.

Liu, Y., 2013. A new extrapolation cascadic multi-grid method for semi-linear PDEs. Journal of Computational Information Systems, (9)11: 4291-4297.

Romberg, W., 1955. Vereinfachte numerische integration. Norske Vid. Selsk. Forhandlinger (Trondhein) 28: 30-36.

Stetter, H.J., 1965. Asymptotic expansions for the error of discretizion algorithms for non-linear functional equations. Numer. Math., 7: 18-31.

Stetter, H.J., 1973. *Analysis of Discretization Methods for Ordinary Differential Equations*. Springer-Verlag, Berlin.