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On Subclass of Harmonic Univalent Function Defined by Ruscheweyh Derivative

[1]Abdul Rhman S. Juma, [2]Hassn H Ebrahim and [3]Shamil I Ahmed

[1]*Department of Mathematics, Alanbar University. Ramadi-Iraq*
[2]*Department of Mathematics, Tikrit University. Tikrit–Iraq*

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ABSTRACT

In the present paper, we introduce a new subclass of harmonic function in the unit disc U by the Ruscheweyh derivative. Also, we obtain coefficient condition, convolution condition, convexcombinations, extreme point,δ-neighborhood, integral operator.

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INTRODUCTION

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain C if both u and v are real harmonic in C . In any simply connected domain $D \subset C$, we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense –preserving in D is that $|h'(z)| > |g'(z)|$ in D, see (Clunie, 1984). In 1984, Clunie and Sheil-Small (Clunie, 1984) investigated the class AH_S and studied some sufficient bounds.Since then the-re have been several paper published related to AH_S and its subclasses. In fact by introducing new subclasses Sheil-small (Sheil-Small, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999), Jahangir (Jahangiri, 1999) and Ahuja (Ahuja, 2005) presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Duren (Duren, 2004), ponnusamy (Ponnusamy, 2007) and ref-erences there in for basic results on the subject.

Denoted by AH_S , the class of function $f = h + \bar{g}$ that are harmonic, univalent and sense-preser-ving in the unit disk $U = \{Z: |Z| < 1\}$ whith normalization $f(0) = h(0) = f_z(0) - 1 = 0$.Then for $f = h + \overline{g} \in AH_S$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k , g(Z) = z + \sum_{k=1}^{\infty} b_k z^k , |b_1| < 1. \quad (1)$$

Observe that AH_S reduces to , the class of normalized univalent functions, if the co- analytic part of f is zero. Also, denoted by $AH_S{}^*$ the subclass of AH_S , consisting of function f that map U onto a starlike domain.

For $f = h + \bar{g}$ given by (1) and $D^\lambda f(z)$ is the Ruscheweyh derivative of f and defined by

$$D^\lambda f(z) = \sum_{k=1}^{\infty} B_k(\lambda) c_k \ z^k , \lambda > -1 , B_k(\lambda) = \frac{(\lambda+1)(\lambda+2) \ldots\ldots (\lambda+n-1)}{(n-1)!}$$

also $D^\lambda f(z) = D^\lambda h(z) + \overline{D^\lambda g(z)}$ (2)

Recently Rosy et al. (Rosy, 2001) defined the subclass $G_S \subseteq AH_S$ consisting of a harmonic univalent function $f(z)$ satisfying the condition

Corresponding Author: Abdul Rhman S. Juma, Department of Mathematics, Alanbar University. Ramadi-Iraq
E-mail: dr_juma@hotmail.com

$$Re\left\{(1+e^{i\alpha})\frac{zf'(z)}{f(z)}-e^{i\alpha}\right\} \geq \gamma \,, 0 \leq \gamma < 1, \alpha \in R.$$

They proved that if $f = h + \overline{g}$ is given by (1) and if

$$\sum_{k=1}^{\infty} \frac{(2k-1-\gamma)}{(1-\gamma)}|a_k| + \frac{(2k+1+\gamma)}{(1-\gamma)}|a_k| \leq 2 \,, 0 \leq \gamma < 1, \quad (3)$$

then f is in $G_S(\gamma)$.

This condition is proved to be also necessary by Rosy ef al. if h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty}|a_k|z^k \,, g(z) = -\sum_{k=1}^{\infty}|b_k|z^k \quad (4)$$

Motivated by this aforementioned work, new we introduce the class $G_S(\lambda, \alpha.p)$ as the subclass of functions of the form (1) satisfy the following condition

$$Re\left\{(1+pe^{i\alpha})\frac{D^{\lambda+q}(z)}{D^{\lambda}f(z)}-pe^{i\alpha}\right\} \geq \gamma \,, 0 \leq \gamma < 1, \alpha \in R \; p \geq 0, q \in \mathbb{N}, \quad (5)$$

where $D^{\lambda}f(z)$ is defied by (2)

Let $\overline{G_S}(\lambda, \alpha, p)$ denoted that the subclass of $G_S(\lambda, \alpha, p)$ which consists of harmonic function $f_k = h + \bar{g}_k$ such that h and g are the form

$$h(z) = z - \sum_{k=2}^{\infty}|a_k|z^k \,, g_n(z) = (-1)^n \sum_{k=1}^{\infty}|b_k|z^k \,. \quad (6)$$

In this paper, we will give sufficient condition for function $f = h + \overline{g}$,where h and g are given by (1) ,to be in the class $G_S(\lambda, \alpha, p)$ it is shown that is coefficient condition is also necessary for function in the class $\overline{G_S}(\lambda, \alpha, p)$ .Also we obtain distortion theorem and characterize the extreme point and convolution conditions for functions in $\overline{G_S}(\lambda, \alpha, p)$.

Closure theorems and application of neighborhood also obtain.

Coefficient Inequality:

We being with a sufficient condition for in $G_S(\lambda, \alpha, p)$.

Theorem 2.1: Let $f = h + \overline{g}$ be given by (2.1). If

$$\sum_{k=1}^{\infty}[\{k(1+\rho)-(\alpha+\rho\}|a_k| + \{k(1+\rho)+(\alpha+\rho\}|b_k|] \, B_K(\lambda) \leq 2(1-\alpha), \quad (7)$$

where $a_1 = 1, \lambda \in \mathbb{N}_0, B_k(\lambda) = \dfrac{(\lambda+1)(\lambda+2)\ldots\ldots(\lambda+n-1)}{(n-1)!}$, $\rho \geq 0$ and $0 \leq \alpha < 1$, $then$ $f is$ sense $-$

Preserving harmonic in U and $f \in G_S(\lambda, \alpha, p)$.

Proof: If $Z_1 \neq Z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right|$$

$$\geq 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=1}^{\infty} k|a_k|} \quad (8)$$

$$\geq 1 - \frac{\dfrac{\sum_{k=1}^{\infty} [\{k(1+\rho)+(\alpha+\rho\}]B_K(\lambda)|b_k|}{1-\alpha}}{1 - \dfrac{\sum_{k=2}^{\infty} [\{k(1+\rho)+(\alpha+\rho\}]\, B_K(\lambda)|a_k|}{1-\alpha}}$$

$$\geq 0 \,,$$

which proves univalence. Not that f is sense preserving in U.This is because

$$|h'(z)| \geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$
$$\geq 1-\sum_{k=2}^{\infty} \frac{\{k(1+\rho)-(\alpha+\rho\}B_k(\lambda)|a_k|}{1-\alpha}$$
$$\geq 1-\sum_{k=1}^{\infty} \frac{\{k(1+\rho)+(\alpha+\rho\}B_k(\lambda)|b_k|}{1-\alpha} \tag{9}$$
$$\geq 1-\sum_{k=1}^{\infty} \frac{\{k(1+\rho)+(\alpha+\rho\}B_k(\lambda)|b_k||Z|^{k-1}}{1-\alpha} \geq \sum_{k=2}^{\infty} k|a_k||z|^{k-1}$$
$$\geq |g'(z)|.$$

Using the fact that$Re\ w > \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ it suffice to show that

$$\left|(1-\alpha)+(1+\rho e^{ir})\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-\rho e^{ir}\right| - \left|(1+\alpha)-(1+\rho e^{ir})\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-\rho e^{ir}\right| \geq 0\,. \tag{10}$$

Substituting the value of $D^{\lambda}f(z)$ in (10) yields, by(7),

$$\left|(1-\alpha-\rho e^{ir})D^{\lambda}f(z)+(1+\rho e^{ir})D^{\lambda+1}f(z)\right| - \left|-(1+\alpha+\rho e^{ir})D^{\lambda}f(z)+(1+\rho e^{ir})D^{\lambda+1}f(z)\right|$$
$$= \left|(2-\alpha)z+\sum_{k=2}^{\infty}\{k(1+\rho e^{ir})+(1-\alpha-\rho e^{ir})\}B_k(\lambda)\times a_k z^k - (-1)^k\sum_{k=1}^{\infty}\{k(1+\rho e^{ir})-(1-\alpha-\rho e^{ir})\}B_k(\lambda)\, b_k z^k\right|$$
$$- \left|-\alpha z+\sum_{k=2}^{\infty}\{k(1+\rho e^{ir})+(1+\alpha+\rho e^{ir})\}B_k(\lambda)\, a_k z^k - (-1)^k\sum_{k=1}^{\infty}\{k(1+\rho e^{ir})+(1+\alpha+\rho e^{ir})\}B_k(\lambda)\, \overline{b_k z^k}\right|$$
$$\geq 2(1-\alpha)|z|\left[1-\sum_{k=2}^{\infty}\frac{\{k(1+\rho)-(\alpha+\rho)\}B_k(\lambda)|a_k||z|^k}{1-\alpha}-\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}B_k(\lambda)|b_k||z|^k}{1-\alpha}\right]$$
$$\geq 2(1-\alpha)|z|\left[1-\sum_{k=2}^{\infty}\frac{\{k(1+\rho)-(\alpha+\rho)\}B_k(\lambda)|a_k|}{1-\alpha}-\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+(\alpha+\rho)\}B_k(\lambda)|b_k|}{1-\alpha}\right].$$

This last expression is non-negative by (8) ,and so the proof is complete.■

The harmonic function

$$f(z) = z+\sum_{k=2}^{\infty}\frac{1-\alpha}{\{k(1+\rho)-(\alpha+\rho)\}B_k(\lambda)}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{\{k(1+\rho)+(\alpha+\rho)\}B_k(\lambda)}\overline{y_k z^k} \tag{11}$$

$where\ \lambda \in \mathbb{N}_0, 0 \leq \rho \leq 1 and \sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 1$, shows that the coefficient bound given by (7) is sharp. The functions of the form (11) are in $G_S(\lambda,\alpha,p)$because

$$\sum_{k=1}^{\infty}\left[\frac{k(1+\rho)-(\alpha+\rho)}{1-\alpha}|a_k|+\frac{k(1+\rho)+(\alpha+\rho)}{1-\alpha}|b_k|\right]B_k(\lambda) \tag{12}$$
$$= 1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k| = 2.$$

In the following theorem, it is shown that the condition (7) is also necessary for functions $f_n = h+\overline{g_n}$, where $h\ and\ g_n\ are\ of\ the\ form$ (6).

Theorem 2.2: Let $f_n = h+\overline{g_n}$ be given by (7).Then $f_n \in \overline{G_S}(\lambda,\alpha,p)$ if and only if

$$\sum_{k=1}^{\infty}[\{k(1+\rho)-(\alpha+\rho\}|a_k|+\{k(1+\rho)+(\alpha+\rho\}|b_k|]\,B_K(\lambda) \leq 2(1-\alpha) \tag{13}$$
$$where\ a_1 = 1, \lambda \in \mathbb{N}_0, B_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\ldots.(\lambda+n-1)}{(n-1)!}, \rho \geq 0\ and\ 0 \leq \alpha < 1.$$

Proof: Since $\overline{G_S}(\lambda,\alpha,p) \subset G_S(\lambda,\alpha,p)$ we only need to prove the "only if" part of Theorem (2.2). To this end, for functions f_n of the form (6) , we notice that the condition (5) is equation to

$$Re\left\{(1+pe^{i\alpha})\frac{D^{\lambda+q}(z)}{D^{\lambda}f(z)}-(pe^{i\alpha}+\alpha\right\} \geq 0$$
$$\Longrightarrow$$

$$Re\ \frac{\{(1+\rho e^{ir})D^{\lambda+1}f(z)-(\rho e^{ir}+\alpha)D^{\lambda}f(z)\}}{D^{\lambda}f(z)}\geq 0$$

$\Longrightarrow$

$$Re\left\{\frac{(1+\rho e^{ir})(z-\sum_{k=2}^{\infty} kB_k(\lambda)|a_k|z^k+\sum_{k=1}^{\infty}(-1)^{2\lambda+1}k|b_k|B_k(\lambda)\bar{z}^k}{z-\sum_{k=2}^{\infty} B_k(\lambda)|a_k|z^k+(-1)^{2\lambda}\sum_{k=1}^{\infty}|b_k|B_k(\lambda)\bar{z}^k}\right.$$

$$\left.-\frac{(\rho e^{ir}+\alpha)(z-\sum_{k=2}^{\infty} kB_k(\lambda)|a_k|z^k+(-1)^{2\lambda+1}\sum_{k=1}^{\infty}k|b_k|B_k(\lambda)\bar{z}^k}{z-\sum_{k=2}^{\infty} B_k(\lambda)|a_k|z^k+(-1)^{2\lambda}\sum_{k=1}^{\infty}|b_k|B_k(\lambda)\bar{z}^k}\right\}\geq 0$$

$\Longrightarrow$

$$Re\left\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}[k(1+\rho e^{ir})-(\rho e^{ir}+\alpha)]B_k(\lambda)|a_k|z^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|z^{k-1}+\frac{\bar{z}}{z}(-1)^{2\lambda}\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\bar{z}^{k-1}}\right.$$

$$\left.-\frac{\frac{\bar{z}}{z}(-1)^{2\lambda}\sum_{k=2}^{\infty}[k(1+\rho e^{ir})+(\rho e^{ir}+\alpha)]B_k(\lambda)|b_k|\bar{z}^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|z^{k-1}+\frac{\bar{z}}{z}(-1)^{2\lambda}\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\bar{z}^{k-1}}\right\}\geq 0 \qquad (14)$$

The above condition (14) must hold for all values of z on the positive real axes, where, $0\leq|z|=\gamma<1$, we must have

$$Re\left\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}(k-\alpha)B_k(\lambda)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}+(-1)^{2\lambda}\sum_{k=1}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}}\right.$$

$$-\frac{(-1)^{2\lambda}\sum_{k=2}^{\infty}(k+\alpha)B_k(\lambda)|b_k|\gamma^{k-1}-\rho e^{ir}\sum_{k=2}^{\infty}(k-1)B_k(\lambda)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}+(-1)^{2\lambda}\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\gamma^{k-1}}-$$

$$\left.-\frac{(-1)^{2\lambda}\rho e^{ir}\sum_{k=2}^{\infty}(k+1)B_k(\lambda)|b_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}+(-1)^{2\lambda}\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\gamma^{k-1}}\right\}\geq 0.$$

Since $\mathrm{Re}(-e^{ir})\geq-|e^{ir}|=-1$, the above inequality reduce to

$$-\frac{(1-\alpha)-\sum_{k=2}^{\infty}\{k(1+\rho)-(\rho+\alpha)\}B_k(\lambda)|a_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\gamma^{k-1}}$$

Australian Journal of Basic and Applied Sciences, 9(5) March 2015, Pages: 307-314

$$-\frac{\sum_{k=2}^{\infty}\{k(1+\rho)+(\rho+\alpha)\}B_k(\lambda)|b_k|\gamma^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|\gamma^{k-1}+\sum_{k=1}^{\infty}B_k(\lambda)|b_k|\gamma^{k-1}} \geq 0. \tag{15}$$

If the condition (13) does not hold, then the numerator in (15) is negative for γ sufficiently close to 1.Hence there exists a $z_0 = y_0$in (0, 1) for which the quotient in (15) is negative. This contrad- icts the condition for $f \in \overline{G_s}\,(\lambda,\alpha,p)$ and so proof is complete.

3.Distortion Bounds:

In this section, we will obtain distortion bounds for function in $\overline{G_S}(\lambda,\alpha,p)$.

Theorem 3.1: Let $f_n \in \overline{G_S}(\lambda,\alpha,p)$ .Then for$|z| = \gamma < 1$, we have

$$|f_n(z)| \leq (1+|b_1|)\gamma + \frac{(1-\alpha)}{[2(1+\rho)-(\rho+\alpha)](1+\lambda)}\left[1-\frac{1+2\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

$$|f_n(z)| \leq (1+|b_1|)\gamma - \frac{(1-\alpha)}{[2(1+\rho)-(\rho+\alpha)](1+\lambda)}\left[1-\frac{1+2\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

Proof: We only prove the lift –hand inequality .The proof for the right inequality is similar and is thus omitted. Let $f_n \in \overline{G_S}(\lambda,\alpha,p)$.Taking the absolute value of f_n , we obtain

$$|f_n(z)|$$

$$= \left|z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{2n}\sum_{k=1}^{\infty} b_k \bar{z}^k\right|$$

$$\leq (1+|b_1|)\gamma + \sum_{k=2}^{\infty}(|a_k|+|b_k|)\gamma^k$$

$$\leq (1+|b_1|)\gamma + \sum_{k=2}^{\infty}(|a_k|+|b_k|)\gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)(\lambda+1)} \times \sum_{k=2}^{\infty}\left(\frac{2(1+\rho)-(\rho+\alpha)(\lambda+1)}{1-\alpha}|a_k| + \frac{2(1+\rho)-(\rho+\alpha)(\lambda+1)}{1-\alpha}|b_k|\right)\gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)(\lambda+1)} \times \sum_{k=2}^{\infty}\left[\frac{k(1+\rho)-(\rho+\alpha)B_k(\lambda)}{1-\alpha}|a_k| + \frac{k(1+\rho)+(\rho+\alpha)B_k(\lambda)}{1-\alpha}|b_k|\right] + \gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)(\lambda+1)}\left[1-\frac{(1+\rho)+(\rho+\alpha)}{1-\alpha}|b_1|\right]\gamma^2$$

$$\leq (1+|b_1|)\gamma + \frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)(\lambda+1)}\left[1-\frac{2+\rho+\alpha}{1-\alpha}|b_1|\right]\gamma^2.$$

The functions

$$f(z) = z + |b_1|\bar{z} + \frac{1}{\lambda+1}\left[\frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)} - \frac{2+\rho+\alpha}{2(1+\rho)-(\rho+\alpha)}\right]\bar{z}^2$$

$$f(z) = (1-|b_1|)z - \frac{1}{\lambda+1}\left[\frac{1-\alpha}{2(1+\rho)-(\rho+\alpha)} - \frac{2+\rho+\alpha}{2(1+\rho)-(\rho+\alpha)}\right]z^2$$

for $|b_1| \leq \frac{1-\alpha}{1+2\rho+\alpha}$ Show that the bounds given in the Theorem 3.1 are sharp

The following covering result follows form the left -hand inequality in Theorem 3.1.

4 .Convex Combination And Extreme Points:

Let the function $f_{n,j}\,(z)$ be defined, for j=1, 2 …m, by

$f_{n,j}(z) = z - \sum_{k=2}^{\infty}|a_{k,j}|z^k + (-1)^n \sum_{k=1}^{\infty}|b_{k,j}|\bar{z}^k$. (16)

Theorem 4.1: Let the function $f_{n,j}(z)$ defined by (16) be in the class $\overline{G_S}(\lambda,\alpha,p)$ for every j=1, 2...m. Then the functions $t_j(z)$defined by

$t_j(z) = \sum_{j=1}^{m} c_j f_{n,j}(z), o \le c_j < 1$, (17)

are also in the class$\overline{G_S}(\lambda,\alpha,p)$, where $\sum_{j=1}^{m} c_j = 1$.

Proof: A cording to the definition of t_j , we can write

$$t_j(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j\,|a_{k,j}|\right)z^k + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m}|b_{k,j}|\right)\bar{z}^k.$$

Further, since $f_{n,j}(z)$ are in$\overline{G_S}(\lambda,\alpha,p)$ for every j=1, 2….m, then

$$\sum_{k=1}^{\infty}\left\{\left[\big(k(1+\rho)-(\alpha+\rho)\big)\left(\sum_{j=1}^{m} c_j\,|a_{k,j}|\right)+\big(k(1+\rho)+(\alpha+\rho)\big)\left(\sum_{j=1}^{m} c_j\,|b_{k,j}|\right)\right]B_k(\lambda)\right\}$$

$$= \sum_{j=1}^{m} c_j\left(\sum_{k=1}^{\infty}[k(1+\rho)-(\alpha+\rho)\,|a_{k,j}| + \big(k(1+\rho)+(\alpha+\rho)\big)|b_{k,j}|]B_k(\lambda)\right)$$

$$\le \sum_{j=1}^{m} c_j\; 2(1-\alpha) \le (1-\alpha).\ \text{Hence theorem 4.1 follows}.$$

Corollary 4.2: The class $\overline{G_S}(\lambda,\alpha,p)$ is closed under convex liner combinations.

Proof: Let the functions $f_{n,j}(z)$ = (j=1, 2…., m) defined by (16) be in the class $\overline{G_S}(\lambda,\alpha,p)$. Then the function $\Psi(z)$ defined by

$\Psi(z) = \mu f_{n,j}(z) + (1-\mu)f_{n,j}(z)\,, o \le \mu < 1$

is in the class $\overline{G_S}(\lambda,\alpha,p)$.

Next we determine the extreme point of closed convex hulls of $\overline{G_S}(\lambda,\alpha,p)$, denoted by clco$\overline{G_S}(\lambda,\alpha,p)$.

Theorem 4.3:Let f_n be given by (7) .Then $f_n \in \overline{G_S}(\lambda,\alpha,p)$ if and only if

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_n h_k(z) + Y_n g_{n,k}(z)\right),$$

where

$$h_1(z) = z\,, h_k(z) = z - \left(\frac{1-\alpha}{k(1+\rho)-(\alpha+\rho)B_k(\lambda)}\right)z^k\ , z = 2,3, \dots \dots,$$

$$g_{n,k}(z) = z - (-1)^n\left(\frac{1-\alpha}{k(1+\rho)+(\alpha+\rho)B_k(\lambda)}\right)\bar{z}^k$$

$and \sum_{k=1}^{\infty}(X_K + Y_K) = 1, X_K \ge 0, Y_K \ge 0$. In particular, the extreme points of$\overline{G_S}(\lambda,\alpha,p)$are $\{h_k\}\ and\ \{g_{nk}\}$.

Proof: For the function f_n of the form (4.3), we have

$$f_n(z) = \sum_{k=1}^{\infty}\left(X_n h_k(z) + Y_n g_{n,k}(z)\right)$$

$$f_n(z) = \sum_{k=1}^{\infty}(X_n + Y_n)z - \sum_{k=1}^{\infty}\frac{1-\alpha}{k(1+\rho)-(\alpha+\rho)B_k(\lambda)}X_n z^k + (-1)^n\sum_{k=1}^{\infty}\frac{1-\alpha}{k(1+\rho)+(\alpha+\rho)B_k(\lambda)}Y_n\bar{z}^k.$$

Then

$\sum_{k=1}^{\infty}\frac{k(1+\rho)-(\rho+\alpha)B_k(\lambda)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{k(1+\rho)+(\rho+\alpha)B_k(\lambda)}{1-\alpha}|b_k|$ (18)

$$= \sum_{k=1}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k = 1 - X_1 \le 1\,,$$

and so $f_n \in clco\ \overline{G_S}(\lambda,\alpha,p)$.

Conversely, suppose that $f_n \in clco\ \overline{G_S}(\lambda,\alpha,p)$. Setting

$X_k = \frac{k(1+\rho)-(\rho+\alpha)B_k(\lambda)}{1-\alpha}|a_k|\,, 0 \le X_k < 1, k = 2,3,$ (19)

$$Y_k = \frac{k(1+\rho)-(\rho+\alpha)B_k(\lambda)}{1-\alpha}|b_k|\,, 0 \le Y_k < 1, k = 2,3, \dots,$$

and $X_1 = 1-\sum_{k=1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$ then f_n can be written as

$$f_n(z) = z - \sum_{k=2}^{\infty}|a_k|z^k + (-1)^n\sum_{k=1}^{\infty}|b_k|\bar{z}^k$$

$$= z - \sum_{k=1}^{\infty}\frac{(1-\alpha)X_k}{k(1+\rho)-(\alpha+\rho)B_k(\lambda)}z^k + (-1)^n\sum_{k=1}^{\infty}\frac{(1-\alpha)Y_k}{k(1+\rho)+(\alpha+\rho)B_k(\lambda)}\bar{z}^k$$

$$= z + \sum_{k=2}^{\infty}(h(z)-z)X_k + \sum_{k=1}^{\infty}\left(g_{n,k}(z)-z\right)Y_k$$

$$= \sum_{k=2}^{\infty}h_k(z)X_k + \sum_{k=1}^{\infty}g_{n,k}(z)Y_k + z(1-\sum_{k=1}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k$$

$$= \sum_{k=2}^{\infty}\left(h_k(z)X_k + g_{n,k}(z)Y_k\right), as\ required\ . (20)$$

Using corollary (4.2) we have clco $\overline{G_S}(\lambda,\alpha,p) = \overline{G_S}(\lambda,\alpha,p)$ .Then the statement of Theorem (4.3) is true for $f \in \overline{G_S}(\lambda,\alpha,p)$.

5 . Neighborhoods:

The $\delta-$ neighborhood $N_\delta(f)$ of f is the set (see (Atintas, 2000) and (Rucsheweyh, 1981)):

$$N_\delta(f) = \{F: \textstyle\sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_K| + |b_1 - B_1| \le \delta\}, \quad (21)$$

where the function $F(z)$ is given by

$$F(z) = z + \textstyle\sum_{k=2}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k} \quad (22)$$

In [8], ÖZTURK and YALCIN .defined the generalized δ-neighbrohood of f to be the set:

$$N_\delta(f) = \left\{F: \sum_{k=2}^{\infty}(k-\alpha)(|a_k - A_k| + |b_k - B_K| + (1-\alpha)|b_1 - B_1| \le \delta(1-\alpha)\right\}.$$

Theorem 5.1: Let $f(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty}(a_k + \overline{b_k z^k})$ be a member of $\overline{G_S}(\lambda,\alpha,p)$. If $\delta \le \frac{2\rho(1-\alpha)}{\rho+1} + |b_1|$, then $N\ (f) \subset \overline{G_S}(\lambda,\alpha,p)$.

Proof: Let $f \in \overline{G_S}(\lambda,\alpha,p)$

$$F(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty}(A_k z^k + \overline{B_k z^k})$$

belong to $N\ (f)$.We have

$$(1-\alpha)|B_1| + \sum_{k=1}^{\infty}(k-\alpha)(|A_k| + |B_K|)$$

$$\le (1-\alpha)|B_1 - b_1| + \sum_{k=1}^{\infty}(k-\alpha)(|A_k - a_k| + |B_K - b_k|) + (1-\alpha)|b_1| + \sum_{k=1}^{\infty}(k-\alpha)(|a_k| + |b_K|)$$

$$\le (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{\rho+1}\sum_{k=1}^{\infty}\{((k(1+\rho)-(\alpha+\rho)+(1+\rho)+(\alpha+\rho)\}B_k(\lambda)(|a_k| + |b_k|)$$

$$\le (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{2(1-\alpha)}{\rho+1} \le 1-\alpha,$$

if $\delta \le \frac{2\rho(1-\alpha)}{\rho+1} + |b_1|$.

Thus $F \in \overline{G_S}(\lambda,\alpha,p)$.

6. Integral Operators:

Now, we examine a closure property of class $\overline{G_S}(\lambda,\alpha,p)$ under the generalized Bernurdi-Libera-Livingston integral operator $L_c(f)$ which is defined by

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt\ , c > -1\ .$$

Theorem 6.1: Let $f \in \overline{G_S}(\lambda,\alpha,p)$. Then $L_c(f) \in \overline{G_S}(\lambda,\alpha,p)$

Australian Journal of Basic and Applied Sciences, 9(5) March 2015, Pages: 307-314

Proof: From the representation of $L_c\big(f\ (z)\big)$, it follows that

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(z) + \overline{g(t)}\right]dt\,.$$

$$= \frac{c+1}{z^c}\int_0^z t^{c-1}\,(t - \sum_{k=2}^{\infty} a_k t^k)\,dt - \overline{\int_0^z t^{c-1}(t - \sum_{k=1}^{\infty} b_k t^k)\ dt}\,.$$

$$= z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k$$

$$where\ A_k = \frac{c+1}{k+1}a_k\ ;\ B_k = \frac{c+1}{k+1}b_k.$$

Therefore,

$$\sum_{k=1}^{\infty}\left(\frac{k(1+\rho)-(\rho+\alpha)}{1-\alpha}\left(\frac{c+1}{k+1}\right)|a_k| + \frac{k(1+\rho)+(\rho+\alpha)}{1-\alpha}\left(\frac{c+1}{k+1}\right)|b_k|\right)B_k(\lambda)$$

$$\leq \sum_{k=1}^{\infty}\left(\frac{k(1+\rho)-(\rho+\alpha)}{1-\alpha}|a_k| + \frac{k(1+\rho)+(\rho+\alpha)}{1-\alpha}|b_k|\right)B_k(\lambda)$$

$\leq 2(1-\alpha)$. Since $f \in \overline{G_S}(\lambda,\alpha,p)$, therefor by Theorem (2.2), $L_c(f) \in \overline{G_S}(\lambda,\alpha,p)$.

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