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Managerial Occurrences Of Ordered Patterns In Rectangular Space Filling Curve

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ABSTRACT

Finite Words corresponding to finite approximations of Rectangular Space Filling Curve are formed. By ordering alphabets of the finite words, relation between number of rises and descents is analyzed. Moreover, the numbers of occurrences of ordered patterns in these words are investigated.

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INTRODUCTION

The use of Space Filling Curves (SFC) has found in many applications such as Routing system, parallel computing, image processing and data bases. The goal of the research presented in this paper is to extend the concept of Space Filling Curves on square frame to Space Filling Curves on rectangular frame. Just as SFCs are convoluted lines that fill a square, these SFCs are carefully elaborated to fill a rectangle.

The author has discussed in (Sergey Kitaev, 2004) about occurrences of some patterns, subsequences and sub words in sigma-sequence. Combinatorics on words have been analyzed in (Aldo de Luca, 1999) and (Jean Berstel, Dominique Perrin, 2007). Masood Ahmed and Shahid Bokhari (Masood Ahmed, 2007) explained about Space Filling Surfaces. The author has introduced the notion of Hilbert words in (Patrice Seebold, 2007). In (Patrice Seebold, *et al*., 2004), Counting occurrences of some patterns in Peano words was done. Counting ordered patterns in words generated by morphisms was done in (Sergey Kitaev, *et al*., 2008).

Finite words which correspond to finite approximations of the Rectangular SFC are produced in section 4 after presenting the generation of Rectangular Space Filling Curve by geometrically and through by a grammar. An ordering is given to the alphabets of finite words and discussed about rises and descents in the next section. Finally, the number of occurrences of ordered patterns in the finite words has been executed.

II. Geometric Generation of Rectangular SFC:

A Space Filling Curve maps a 1-dimensional space onto a higher-dimensional space e.g., the unit interval onto the unit square. A geometric generation principle for the extension of Hilbert curve construction to fill a rectangle region is suggested. The method is a work out in recursive thinking and can be summed up in a few lines. Let us assume that the unit interval I can be mapped continuously onto the rectangle $\Omega\left[0,\frac{2}{3}\right]x[0,1]$.

2.1.Initial mapping:

If I is partitioned into six congruent subintervals then it should be possible to partition Ω into six congruent sub squares, such that each subinterval will be mapped continuously onto one of the sub squares.

2.2. Iteration mapping:

If the subintervals are partitioned into nine congruent subintervals then it should be possible to partition the sub squares into nine congruent sub squares, such that each subinterval will be mapped continuously onto one of the sub squares. This reasoning can be repeated by again partitioning each subinterval into nine congruent subintervals and doing the same for the respective sub squares. When repeating this procedure make sure that the sub squares are arranged in such a way that adjacent sub squares correspond to adjacent subintervals. Like this the overall continuity of the mapping is preserved. If an interval corresponds to a square, then its subintervals must correspond to the sub squares of

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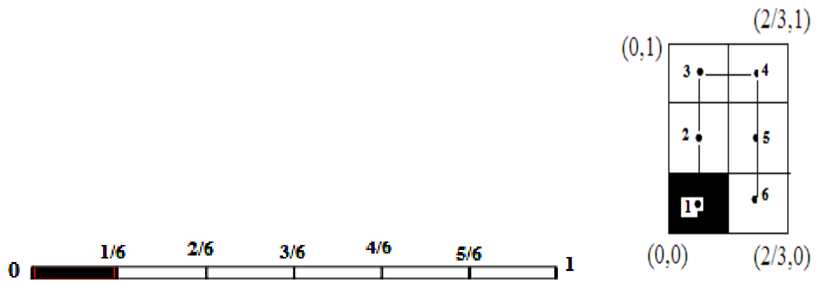
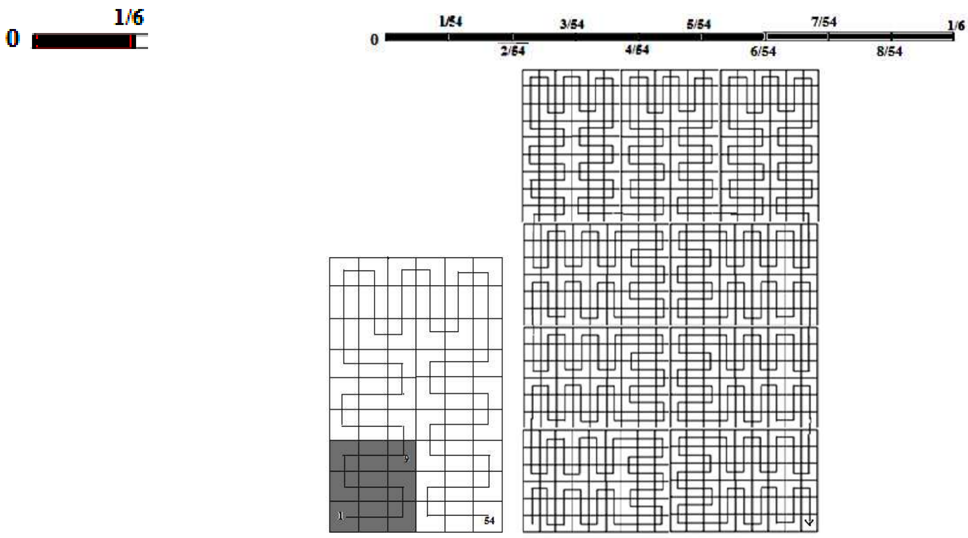
that square. This inclusion relationship assures that a mapping of the n^{th} iteration preserves the mapping of the $(n\text{-}1)^{th}$ iteration.

2.3.Initial Iteration:



First Iteration:



First iteration **Second iteration**

III. Representation of the Rectangular SFC Through a Grammar:

In the construction of this Hilbert Rectangular curve, the four templates H, A, B and C illustrated in the figure (1) are used in every iterations. These templates are substituted in every iteration step into a first iteration of this Space Filling Curve. These fixed substitution procedure can be described by a grammar G = (V,T,P,S) where V = $\{H, A, B, C\}$, T = $\{\uparrow, \downarrow, \rightarrow, \leftarrow\}$, S = $\{H\}$ and P is defined by

$H \rightarrow [C \uparrow C \uparrow C \uparrow H \rightarrow H \rightarrow H \downarrow A \downarrow A \downarrow A]$,

$A \rightarrow [B \leftarrow B \leftarrow B \leftarrow A \downarrow A \downarrow A \rightarrow H \rightarrow H \rightarrow H]$

$C \rightarrow [H \rightarrow H \rightarrow H \rightarrow C \uparrow C \uparrow C \leftarrow B \leftarrow B \leftarrow B]$

$B \rightarrow [A \downarrow A \downarrow A \downarrow B \leftarrow B \leftarrow B \uparrow C \uparrow C \uparrow C]$

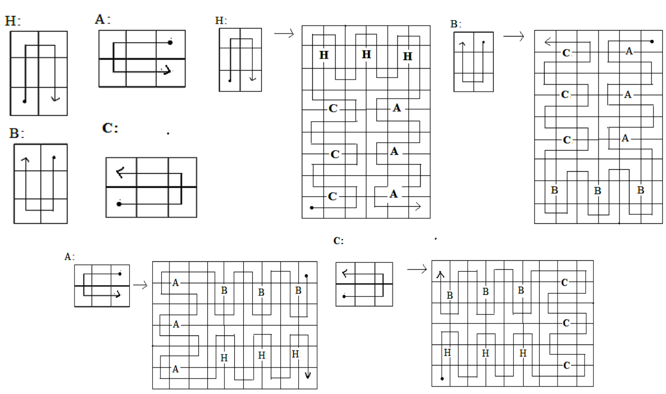


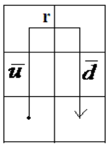
Fig. 1:

IV. Finite Word H_n For The Rectangular SFC:

The construction of the Hilbert Rectangular curve is obtained by drawing, without removing the pen from the surface of the paper to fill the rectangle of size 3^nx2^n, an infinite succession of unit lines and double unit lines left, right, up or down. Thus this succession can be represented by a word over the alphabet $\sum = \{u, d, r, l, \bar{u}, \bar{d}, \bar{r}, \bar{l}\}$ where u stands for unit line up, d stands for unit line down, r stands for unit line right, ℓ stands for unit line left, $\bar{u}$ stands for double units line up, $\bar{d}$ stands for double units line down, $\bar{r}$ stands for double units line right, $\bar{\ell}$ stands for double units line left. Let us explain the algorithm of rectangular Hilbert Curve. The general idea is to divide, at step n, the rectangle into $6(9)^{n-1}$ equal sub squares each of them containing an equal length part of the curve (except the first and the last ones which contain a part of length ½). The curve so obtained is then depicted by a word of length $[4(9)^{n-1} - 1]$ which is called as the n^{th} rectangular Hilbert word H_n

Therefore

$H_1 = \bar{u}\, r\, \bar{d}$



Similarly H_2 = $\bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}$

H_3 = $\bar{u}\, r\, \bar{d}\, r$ $\bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}$

u

$\bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r$ $\bar{u}\, r\, \bar{d}\, r\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}$

u

$\bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}$

u

$\bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}$

r

$\bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}$

r

$\bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{r}\, u\, \bar{\ell}\, u\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}$

d

$\bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}$

d

$\bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}$

d

$\bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{d}\, \ell\, \bar{u}\, \ell\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, d\, \bar{\ell}\, d\, \bar{r}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}\, r\, \bar{u}\, r\, \bar{d}$

Now let us define on $\sum$ three literal morphisms f, t_ℓ and t_r by

f (u) = d, f (d) = u, f(r) = r, f (ℓ) = ℓ , f ($\bar{u}$) = $\bar{d}$, f ($\bar{d}$) = $\bar{u}$, f($\bar{r}$) = $\bar{r}$, f($\bar{\ell}$) = $\bar{\ell}$

t_ℓ (u) = ℓ , t_ℓ (d) = r, t_ℓ (r) = u, t_ℓ (ℓ) = d, $t_\ell(\bar{u}) = \bar{\ell}$, $t_\ell(\bar{d}) = \bar{r}$, $t_\ell(\bar{r}) = \bar{u}$, $t_\ell(\bar{\ell}) = \bar{d}$

t_r (u) = r, t_r (d) = ℓ t_r (r) = d, t_r (ℓ) = u, $t_r(\bar{u}) = \bar{r}$, $t_r(\bar{d}) = \bar{\ell}$, $t_r(\bar{r}) = \bar{d}$, $t_r(\bar{\ell}) = \bar{u}$

These three morphisms represent a vertical flip, a quarter turn left rotation, and a quarter turn right rotation respectively. Using these morphisms, the recurrence relation for finite word H_n (n ≥ 1) is generated by

$$H_{n+1} = \rho(H_n)u\rho(H_n)u\rho(H_n)uH_n rH_n rH_n d\lambda(H_n)d\lambda(H_n)d\lambda(H_n)$$

(1) where $\rho = t_\ell \circ f$ and $\lambda = t_r \circ f$

Theorem 4.1: For any n ≥ 1,

1. $|H_n| = 4(9)^{n-1} - 1$
2. $|H_n|_{\bar{u}} = |H_n|_{\bar{d}}$ and $|H_n|_{\bar{\ell}} = |H_n|_{\bar{r}}$
3. $|H_{n+1}|_{\bar{u}} = 3|H_n|_{\bar{u}} + 3|H_n|_{\bar{r}} + 3|H_n|_{\bar{\ell}}$
4. $|H_{n+1}|_{\bar{d}} = 3|H_n|_{\bar{d}} + 3|H_n|_{\bar{\ell}} + 3|H_n|_{\bar{r}}$
5. $|H_{n+1}|_{\bar{r}} = 3|H_n|_{\bar{r}} + 3|H_n|_{\bar{u}} + 3|H_n|_{\bar{d}}$
6. $|H_{n+1}|_{\bar{\ell}} = 3|H_n|_{\bar{\ell}} + 3|H_n|_{\bar{d}} + 3|H_n|_{\bar{u}}$
7. $|H_{n+1}|_r = 3|H_n|_r + 3|H_n|_u + 3|H_n|_d + 2$

8. $|H_{n+1}|_u = 3|H_n|_u + 3|H_n|_r + 3|H_n|_\ell + 3$
9. $|H_{n+1}|_d = 3|H_n|_d + 3|H_n|_\ell + 3|H_n|_r + 3$
10. $|H_{n+1}|_\ell = 3|H_n|_\ell + 3|H_n|_d + 3|H_n|_u$
11. $|H_n|_{\bar{u}} = |H_n|_{\bar{d}} = \begin{cases} \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \ n \ is \ even \\ \frac{3^{n-1}(3^{n-1}+1)}{2}, & if \ n \ is \ odd \end{cases}$
12. $|H_n|_{\bar{r}} = |H_n|_{\bar{\ell}} = |H_n|_u = |H_n|_d = \begin{cases} \frac{3^{n-1}(3^{n-1}+1)}{2}, & if \ n \ is \ even \\ \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \ n \ is \ odd \end{cases}$

$|H_n|_u = 0$ means that there is zero occurrence of u in H_n.

13. $|H_n|_r = \begin{cases} \frac{3^n(3^{n-2}+1)}{2} - 1, & if \ n \ is \ odd \\ \frac{3^{n-1}(3^{n-1}+1)}{2} - 1, & if \ n \ is \ even \end{cases}$
14. $|H_n|_\ell = \begin{cases} \frac{3^{n-1}(3^{n-1}-1)}{2}, & if \ n \ is \ odd \\ \frac{3^n(3^{n-2}-1)}{2}, & if \ n \ is \ even \end{cases}$

Proof:

The recurrence relation for $|H_n|$ is given by $|H_{n+1}| = 9|H_n| + 8$, $|H_1| = 3$. Solving this equation with initial condition, the equality (1) is obtained. The equalities from (2) to (10) can be obtained from the definition of ρ, λ and the formation (1) of H_n. Other equalities can be proved by induction on n.

V. Rises and descents in H_n:

Let us order the alphabets of as $\bar{u} < u < r < \bar{r} < \bar{d} < d < \ell < \bar{\ell}$

Proposition 5.1:

Let $x_2, \ldots, x_n$ be non-empty words over $\{\bar{u}, u, r, \bar{r}\}$, $y_1, y_2, \ldots, y_{k-1}$ non-empty words over $\{\bar{d}, d, \ell, \bar{\ell}\}$, x_1 is a word over $\{\bar{u}, u, r, \bar{r}\}$ and y_k is a word over $\{\bar{d}, d, \ell, \bar{\ell}\}$ (may be $x_1 = \epsilon$. Or $y_k = \epsilon$, or both). Let $w = x_1 y_1 x_2 y_2 \ldots x_k y_k$, then

$$R(\rho(w)) = \begin{cases} D(w) - 1, & if \ x_1 = y_k = \varepsilon \\ D(w) + 1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ D(w), & otherwise \end{cases}$$

$$D(\rho(w)) = \begin{cases} R(w) + 1, & if \ x_1 = y_k = \varepsilon \\ D(w) - 1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ R(w), & otherwise \end{cases}$$

$R(\lambda(w)) = D(w)$ and $D(\lambda(w)) = R(w)$ where R (w) and D (w) denote the number of rises and descents respectively in a word w.

Proof:

Clearly $D(w) = \sum_{i=1}^{k} [D(x_i) + D(y_i)] + k - 1$

and $R(w) = \sum_{i=1}^{k} [R(x_i) + R(y_i)] + \begin{cases} k-2, & if \ x_1 = y_k = \varepsilon \\ k, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ k-1, & otherwise \end{cases}$

Let $\rho(w) = x_1' y_1' x_2' y_2' \ldots x_k' y_k'$

From the definition of ρ, $R(x_i') = D(x_i)$, $D(x_i') = R(x_i)$, $R(y_i') = D(y_i)$, $D(y_i') = R(y_i)$,

$$R(\rho(w)) = \sum_{i=1}^{k} [R(x_i') + R(y_i')] + \begin{cases} k-2, & if \ x_1' = y_k' = \varepsilon \\ k, & if \ x_1' \neq \varepsilon \ and \ y_k' \neq \varepsilon \\ k-1, & otherwise \end{cases} =$$

$$\sum_{i=1}^{k} [D(x_i) + D(y_i)] + \begin{cases} k-2, & if \ x_1 = y_k = \varepsilon \\ k, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ k-1, & otherwise \end{cases}$$

This implies that $R(\rho(w)) = \begin{cases} D(w) - 1, & if \ x_1 = y_k = \varepsilon \\ D(w) + 1, & if \ x_1 \neq \varepsilon \ and \ y_k \neq \varepsilon \\ D(w), & otherwise \end{cases}$

Since λ is the literal morphism which inverts the order of the letters, we get $R(\lambda(w)) = D(w)$ and $D(\lambda(w)) = R(w)$

Theorem 5.1:

For any n ∈ N,

$D(H_{2n+2}) = (81)^n 18 - 1$

$D(H_{2n+1}) = 2\{(81)^n - 1\}$

$R(H_{2n+2}) = (81)^n 18 - 1$

$R(H_{2n+1}) = 2(81)^n$.

Proof:

From the construction of H_n,

$D(H_{2n+2}) = 6R(H_{2n+1}) + 3D(H_{2n+1}) + 5$

$D(H_{2n+1}) = 6R(H_{2n}) + 3D(H_{2n}) + 7$

$R(H_{2n+1}) = 3R(H_{2n}) + 6D(H_{2n}) + 9$

$R(H_{2n+2}) = 6D(H_{2n+1}) + 3R(H_{2n+1}) + 11$.

And using the above proposition 5.1, the theorem can be proved.

VI. Occurences Of Ordered Patterns In H_n:

Let us assume here after that the value of nCx is zero for x > n. For any m,n ∈ N - {0}, the number of occurrences of the ordered pattern $\tau = (\bar{u}\#)^m$ or $\tau = (\bar{d}\#)^m$ in H_n is

$\binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m}$ when n is odd , $m \le \frac{3^{n-1}(3^{n-1}+1)}{2}$

This implies that $1 \le m < \frac{(3^{n-1}+1)^2}{2}$

(i.e) $0 \le \log m < 2\log M$ where $M = (3^{n-1}+1)$

and it is $\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m}$ when n is even, $m \le \frac{3^{n-1}(3^{n-1}-1)}{2}$

(i.e) $1 \le m < \frac{(3^{n-1})^2}{2}$ otherwise $0 \le \log m < 0.9542n$

The number of occurrences of the ordered pattern $\tau = (\bar{\ell}\#)^m$ or $\tau = (\bar{r}\#)^m$ or $\tau = (u\#)^m$ or $\tau = (d\#)^m$ in H_n is

$\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m}$ when n is odd , $m \le \frac{3^{n-1}(3^{n-1}-1)}{2}$

(i.e) $1 \le m < \frac{(3^{n-1})^2}{2}$ (or) $0 \le \log m < 0.9542n$

and it is $\binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m}$ when n is even, $m \le \frac{3^{n-1}(3^{n-1}+1)}{2}$

(i.e) $1 \le m < \frac{(3^{n-1}+1)^2}{2}$ otherwise $0 \le \log m < 2\log M$ where $M = (3^{n-1}+1)$

Likewise the number of occurrences of the ordered pattern $\tau = (r\#)^m$ in H_n is

$\binom{\frac{3^{n}(3^{n-2}+1)}{2} - 1}{m}$ when n is odd , $m \le \frac{3^{n}(3^{n-2}+1)}{2} - 1$.

So $1 \le m < \frac{(3^{n})^2}{2}$ (or) $0 \le \log m < 0.9542n$

and it is $\binom{\frac{3^{n-1}(3^{n-1}+1)}{2} - 1}{m}$ when n is even, $m \le \frac{3^{n-1}(3^{n-1}+1)}{2} - 1$

As a result $1 \le m < \frac{(3^{n-1}+1)^2}{2}$ otherwise $0 \le \log m < 2\log M$ where $M = (3^{n-1}+1)$

The number of occurrences of the ordered pattern $\tau = (\ell\#)^m$ in H_n is

$$\binom{\frac{3^n(3^{n-2}-1)}{2}}{m} \text{ when n is odd }, m \le \frac{3^{n-1}(3^{n-1}-1)}{2}$$

(i.e) $1 \le m < \frac{(3^{n-1})^2}{2}$ (or) $0 \le \log m < 0.9542n$

and it is $\binom{\frac{3^n(3^{n-2}-1)}{2}}{m}$ when n is even, $m \le \frac{3^n(3^{n-2}-1)}{2}$

(i.e) $1 \le m < \frac{(3^n)^2}{2}$ (or) $0 \le \log m < 0.9542n$

For the general case, let us have the following theorem.

Theorem 6.1:

For any m,n $\in$ N - {0}, the number of occurrences of the ordered pattern $\tau = (1\#)^m$ in H_n is equal to

$$2\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + 4\binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m} + \binom{\frac{3^{n-1}(3^{n-1}+1)}{2}-1}{m} + \binom{\frac{3^n(3^{n-2}-1)}{2}}{m}$$ when n is an even number, $m \le \frac{3^{n-1}(3^{n-1}+1)}{2}$

(i.e) $1 \le m < \frac{(3^{n-1}+1)^2}{2}$ (or) $0 \le \log m < 2\log M$ where $M = (3^{n-1}+1)$

$$2\binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m} + 5\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + \binom{\frac{3^n(3^{n-2}+1)}{2}-1}{m}$$ when n is an odd number, $m \le \frac{3^n(3^{n-2}+1)}{2} - 1$

Consequently $1 \le m < \frac{(3^n)^2}{2}$ (or) $0 \le \log m < 0.9542n$

Proof:

Let $x \in \Sigma$. The number of occurrences of a subsequence x^m in H_n is obviously given by $\binom{|H_n|_x}{m}$. By applying (11) to (14) of theorem 4.1, the theorem can be proved.

Vii. Counting occurences of $N_\tau(w)$:

Let $N_\tau(w)$ denote the number of occurrences of the Pattern τ in the word w. Using the previous results, the number of occurrences of the patterns $\tau_1(x,y) = [x(\#y)^m$, $\tau_2(x,y) = (x\#)^m y]$ and $\tau_3(x,y,z) = [x(\#y)^m\#z]$ where x,y,z $\in$ {1,2,3} can be counted.

The pattern $\tau_1(1,2) = [1(\#2)^m$, then the letter 1 in this Pattern must correspond to the leftmost letter of the Word H_n. Now if n = 2k + 1, then H_n starts with the letter $\bar{u}$. Therefore $(\#2)^m$ can correspond to any subsequence $(\#i)^m$, where i = u,r, $\bar{r}$, $\bar{d}, d, \ell, \bar{\ell}$. Hence there are

$$S_1 = \binom{\frac{3^n(3^{n-2}+1)}{2}-1}{m} + 5\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + \binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m}, \quad m \le \frac{3^n(3^{n-2}+1)}{2} - 1$$

occurrences of the pattern $\tau_1(1,2)$ in H_n.

This implies that $1 \le m < \frac{(3^n)^2}{2}$, (i.e) $0 \le \log m < 0.9542n$

Using this idea, the following table can be formed.

x	y	$N_{\tau_1(x,y)}(H_{2k+1})$	$N_{\tau_1(x,y)}(H_{2k+2})$	$N_{\tau_2(x,y)}(H_{2k+1})$	$N_{\tau_2(x,y)}(H_{2k+2})$
1	1	S_2	S_2	S_2	S_1
1	2	S_1	S_3	S_5	S_4
2	1	0	S_4	S_6	S_3

$$S_2 = \binom{\frac{3^{n-1}(3^{n-1}+1)}{2} - 1}{m}, \quad m \le \frac{3^{n-1}(3^{n-1}+1)}{2} - 1$$

$$S_3 = \binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + 2\binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m} + \binom{\frac{3^{n}(3^{n-2}-1)}{2}}{m}, \ m \le \frac{3^{n-1}(3^{n-1}+1)}{2}$$

$$S_4 = \binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + \binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m} + \binom{\frac{3^{n-1}(3^{n-1}+1)}{2} - 1}{m}, \ m \le \frac{3^{n-1}(3^{n-1}+1)}{2}$$

$$S_5 = \binom{\frac{3^{n-1}(3^{n-1}+1)}{2}}{m} + 2\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + \binom{\frac{3^{n}(3^{n-2}+1)}{2} - 1}{m}, \ m \le \frac{3^{n}(3^{n-2}-1)}{2} - 1$$

$$S_6 = 3\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m}, \quad m \le \frac{3^{n-1}(3^{n-1}-1)}{2},$$

$$S_7 = \binom{\frac{3^{n-1}(3^{n-1}+1)}{2} - 2}{m}, m \le \frac{3^{n-1}(3^{n-1}+1)}{2} - 2$$

$$S_8 = 2\binom{\frac{3^{n-1}(3^{n-1}-1)}{2}}{m} + \binom{\frac{3^{n}(3^{n-2}+1)}{2} - 1}{m}, \ m \le \frac{3^{n}(3^{n-2}-1)}{2} - 1$$

x	y	z	$N_{\tau_3(x,y,z)}(H_{2k+1})$	$N_{\tau_3(x,y,z)}(H_{2k+2})$
1	1	1	0	S_7
1	1	2	S_2	0
1	2	1	0	S_3
1	2	2	S_2	0
2	1	2	0	S_4
1	2	3	S_8	0
1	3	2	S_6	0

Conclusion:

In this paper, we observed recursive occurrences of rises, descents and ordered patterns in the finite words which represents finite approximations of the Rectangular Space Filling Curve.

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