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On Almost Semi Star Generalized Proper G-spaces

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ABSTRACT

The main purpose of our paper is to introduce special type of proper G-spaces, namely, almost semi star generalized proper G-space. This space is neither a semi star generalized proper G-space nor a proper G-space. We study the properties of this space and we show that the suspension SX of a connected semi star generalized proper G-space is almost semi star generalized proper G-space but neither proper G-space nor semi star generalized proper G-space. Also we study the equivalent definitions, finite product and the equivariant homeomorphic image of almost semi star generalized proper G-spaces.

INTRODUCTION

The concept of group actions is considered one of the very important concepts in geometrical topology, which began at the beginning of the twentieth century, especially after Hilbert presented his fifth problem in his famous list. A proper G-space was first introduced by Palais, R. S. (1961). Also, Levine, N. (1963) and Rao, K. C. and K. Joseph, (2002) introduced the concept of semi-open sets and s*g-open sets in topological spaces. The main goal of our paper is to introduce special type of proper G-spaces, namely, almost semi star generalized proper G-space. This space is neither a proper G-space nor a semi star generalized proper G-space. We study the equivalent definitions and properties of almost semi star generalized proper G-spaces. Moreover we show that the suspension SX of a connected semi star generalized proper G-space is almost semi star generalized proper G-space but neither proper G-space nor semi star generalized proper G-space. Finally, s*g-open sets is also called s*-open sets by Al-Meklafi, S. (2002) and Mahmood, S. I. (2010, 2012; 2015).

1. Preliminaries:

Definition 1.1 (Bredon, G. E., 1972):

Let G be a topological group and X be a topological space. Then we say that G acts on X if there exists a continuous function $\phi : G \times X \to X$ such that:

i) $\phi(e, x) = x$, for each $x \in X$, where e is the identity element of G.

ii) $\phi(gh, x) = \phi(g, \phi(h, x))$, for each $g, h \in G$ and $x \in X$.

ϕ is called an action of G on X. The triple (G, X, ϕ) is called a topological transformation group, and $\phi(g, x)$ will be denoted by $g.x$.

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Definition 1.2 (Bredon, G. E., 1972):

A G-space X is a topological transformation group (G, X, ϕ) where G is a locally compact non-compact toplological group and X is a completely regular Hausdorff space.

Definition 1.3 (Bredon, G. E., 1972):

Let X be a G-space. A subset U of X is called invariant under G iff $G.U = U$, where $G.U = \{g.a : g \in G, u \in U\}$

Definition 1.4 (Palais, R. S. , 1961):

Let X be a G-space. If A and B are subsets of X, then A is said to be thin relative to B if the set $((A, B)) = \{g \in G : g.A \bigcap B \neq \phi\}$ is relatively compact in G. If A is thin relative to itself, then it is called thin.

Definition 1.5 (Palais, R. S. , 1961):

Let X be a G-space. A subset S of X is said to be small iff each point in X has a neighborhood which is thin relative to S.

Definition 1.6 (Palais, R. S. , 1961):

A G-space X is called proper iff each point of X has a small neighborhood.

Definition 1.7 (Al-Srraai, S. J., 2000):

A subset U of a G-space X with $U \neq X$ is called star if for each $x \in X$ there exists $g \in G$ such that $g.x \in U$.

Definition 1.8 (Bredon, G. E., 1972):

Let (G, X_1, ϕ_1) and (G, X_2, ϕ_2) be topological transformation groups. A continuous function $\psi : X_1 \to X_2$ is called an equivariant function if ψ satisfies:

For each $g \in G, x \in X$, $\psi(\phi_1(g, x)) = \phi_2(g, \psi(x))$ simply, $\psi(g.x) = g.\psi(x)$

Definition 1.9 (Levine, N., 1963):

Let (X, τ) be a topological space and $U \subseteq X$. Then U is called semi-open if there is an open subset O of X such that $O \subseteq U \subseteq \overline{O}$. The complement of a semi-open set is defined to be semi-closed.

Definition 1.10 (Levin, N., 1970):

Let (X, τ) be a topological space and $U \subseteq X$. Then U is called a generalized closed (g-closed) if $\overline{U} \subset O$ whenever $U \subset O$ and O is open in X. The complement of a g-closed set is defined to be generalized open (g-open).

Definition 1.11 (Rao, K. C. and K. Joseph, 2002):

Let (X, τ) be a topological space and $U \subseteq X$. Then U is called semi star generalized open (s*g-open) if $F \subseteq U^o$ whenever $F \subseteq U$ and F is semi-closed in X. The complement of a semi star generalized open set is defined to be semi star generalized closed (s*g-closed).

Proposition 1.12 (Tareq, J. S., 2015):

Let (X, τ) be a topological space and $U \subseteq Y \subseteq X$. If Y is an open set in X and U is an s*g-open set in Y, then U is an s*g-open set in X.

Definition 1.13 (Rao, K. C. and K. Joseph, 2002):

A subset N of a topological space (X, τ) is called a semi star generalized neighborhood (s*g-neighborhood) of a point x in X if there exists an s*g-open set O in X such that $x \in O \subseteq N$.

Definition 1.14 (Mahmood, S. I. , 2015):

A topological space (X, τ) is called semi star generalized compact (s*g-compact) if every s*g-open cover of X has a finite subcover.

Definition 1.15:
A topological space (X,τ) is called semi star generalized locally semi star generalized compact (s*g-locally s*g-compact) if each point in X has an s*g-compact s*g-neighborhood.

Every s*g-compact space is an s*g-locally s*g-compact, but the converse is not true in general we see that by the following example:

Example 1.16:
Let X be any infinite set and (X,D) be the discrete topology. Then X is s*g-locally s*g-compact, but is not s*g-compact.

Definition 1.17 (Willard, S., 1970):
The suspension SX of a topological space space (X,τ) is the quotient space obtained from $X\times I$ (where $I=[0,1]$ is the closed interval subspace of the usual topological space (R,μ)) by identifying each of $X\times\{1\}$ and $X\times\{0\}$ to a single point. Each point of SX will be denoted by $[x,k]$ where $x\in X$ and $k\in I$.

2. Semi Star Generalized Proper G-spaces:

Definition 2.1:
Let X be a G-space. A subset S of X is said to be semi star generalized small (s*g-small) iff each point in X has an s*g-neighborhood which is thin relative to S.

Definition 2.2:
A G-space X is called semi star generalized proper (s*g-proper) iff each point of X has an s*g-small s*g-neighborhood.

Examples 2.3:
i) (R,+) with the usual topology is a locally compact non-compact topological group. Also, R with the usual topology is a completely regular Hausdorff space. Then R acts on itself as follows: $r_1.r_2=r_1+r_2$ for each $r_1,r_2\in$R. It is clear that R is an R-space. To prove that R is an s*g-proper R-space. Let x be a point of R, then there is an s*g-small s*g-neighborhood S of x, where $S=(x-\in_1,x+\in_1)$, $\in_1>0$ i.e. for any point y of R, there exists an s*g-neighborhood $U=(y-\in_2,y+\in_2)$, $\in_2>0$ of y which is thin relative to S. That is the set:
$((S,U))=\{t\in R:t+S\cap U\neq\phi\}=(y-x-(\in_1+\in_2),y-x+(\in_1+\in_2))$ is relatively compact in R.
Thus R is an s*g-proper R-space.
ii) (R-{0},.) with the usual topology is a locally compact non-compact topological group. Also, R^2 with the usual topology is a completely regular Hausdorff space. Then R-{0} acts on R^2 as follows:
$r.(x_1,x_2)=(rx_1,rx_2)$ for each $r\in$R-{0} and each $(x_1,x_2)\in R^2$. It is clear that R^2 is an R-{0}-space.
But $(0,0)\in R^2$ has no s*g-small s*g-neighborhood S, since for each s*g-neighborhood U of $(0,0)$, the set $((S,U))=$R-{0} is not relatively compact in R-{0}. Thus R^2 is not an s*g-proper R-{0}-space.

Theorem 2.4:
Every proper G-space is an s*g-proper G-space.

Proof:
It is obvious.

Theorem 2.5:
If X is a G-space. Then:
i) Any subset of an s*g-small set is s*g-small.
ii) If U is an s*g-compact subset of X and S is an s*g-small subset of X, then U is thin relative to S.
iii) Every s*g-compact subset of an s*g-proper G-space X is s*g-small.
iv) Every s*g-compact subset of an s*g-proper G-space X is thin.
v) If U is an s*g-compact subset of an s*g-proper G-space X, then $((U,U))$ is a compact subset of G.
vi) If G is a compact topological group, then any G-space X is s*g-proper.

Proof:

i) Let S_1 be an s*g-small subset of X and $S_2 \subset S_1$. To prove that S_2 is s*g-small. Let $x \in X$, since S_1 is s*g-small, then there exists an s*g-neighborhood U of x which is thin relative to S_1. Since $((S_2, U)) \subseteq ((S_1, U))$, then by (Palais, R. S. , 1961, definition (1.1.1)), $((S_2, U))$ is relatively compact in G. Therefore S_2 is s*g-small.

ii) To prove that U is thin relative to S. Since S is s*g-small, then for each $x \in U$, there exists an s*g-neighborhood U_x of x which is thin relative to S. So there exists an s*g-open set O_x of x such that $x \in O_x \subseteq U_x$. Hence $\{O_x\}_{x \in U}$ is an s*g-open cover of U. But U is s*g-compact, then there is a finite subcover of U, say $\{O_{x_i}\}_{i=1}^n$ such that $U \subseteq \bigcup_{i=1}^n O_{x_i}$. Since $((\bigcup_{i=1}^n O_{x_i}, S)) = \bigcup_{i=1}^n ((O_{x_1}, S))$ and $((O_{x_i}, S))$ is relatively compact in G $(\forall i = 1, \ldots n)$, then so is $((\bigcup_{i=1}^n O_{x_i}, S))$. Since $U \subseteq \bigcup_{i=1}^n O_{x_i}$ and $S \subseteq S$, then by (Palais, R. S. , 1961, definition (1.1.1)), $((U, S))$ is relatively compact in G. Therefore U is thin relative to S.

iii) Let U be an s*g-compact subset of an s*g-proper G-space X. To prove U is s*g-small. Let $x \in X$, since X is s*g-proper, then x has S as an s*g-small s*g-neighborhood. We obtain from (ii) U is thin relative to S. Thus U is s*g-small.

iv) Let U be an s*g-compact subset of an s*g-proper G-space X, then by (iii) U is s*g-small and hence by (ii) U is thin relative to itself. Therefore U is thin.

v) Since U is an s*g-compact subset of X, then by (Mahmood, S.I. , 2015, theorem (2.4)), U is compact. Therefore by (Palais, R. S. , 1961, definition (1.1.1)), we have $((U, U))$ is closed in G. But $((U, U))$ is relatively compact in G (from (iv)), then $((U, U))$ is a compact subset of G.

vi) It is obvious.

Theorem 2.6:

If X is an s*g-locally s*g-compact G-space, then the following are equivalent:

i) X is an s*g-proper G-space.

ii) Every s*g-compact subset of X is s*g-small.

iii) Every s*g-compact subset of X is thin.

Proof:

$(i) \Rightarrow (ii)$. By (iii) of theorem (2.5).

$(ii) \Rightarrow (iii)$. Let U be an s*g-compact subset of X, then by (ii) U is s*g-small. Therefore by theorem (2.5), (ii), U is thin.

$(iii) \Rightarrow (i)$. To prove that X is an s*g-proper G-space. Let $x \in X$, since X is s*g-locally s*g-compact, then x has S as an s*g-compact s*g-neighborhood. To prove that S is s*g-small. Let $y \in X$, since X is s*g-locally s*g-compact, then there exists an s*g-compact s*g-neighborhood U of y. Since $S \bigcup U$ is s*g-compact (by Mahmood, S. I. , 2015, theorem (2.15)), then by (iii) $S \bigcup U$ is thin. Since $U \subseteq S \bigcup U$ and $S \subseteq S \bigcup U$, then $((U, S)) \subseteq ((U \bigcup S, U \bigcup S))$. That is U is thin relative to S, hence S is s*g-small. Thus X is an s*g-proper G-space.

Theorem 2.7:

Let X_1 and X_2 be two G-spaces. Then $X_1 \times X_2$ is an s*g-proper G-space if at least one of them is s*g-proper.

Proof:

Suppose that X_1 is an s*g-proper G-space. By (Al-Jeburi, S. S., 2001) we get $X_1 \times X_2$ is a G-space. To prove that $X_1 \times X_2$ is s*g-proper. Let $(x_1, x_2) \in X_1 \times X_2 \Rightarrow x_1 \in X_1$ and $x_2 \in X_2$. Since X_1 is s*g-proper, then x_1 has S as an s*g-small s*g-neighborhood. By (Mahmood, S. I. , 2015, proposition (1.7)), we get $S \times X_2$ is an

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s*g-neighborhood of (x_1, x_2) in $X_1 \times X_2$. To prove that $S \times X_2$ is s*g-small. For each $(y_1, y_2) \in X_1 \times X_2$ $\Rightarrow y_1 \in X_1$ and $y_2 \in X_2$. Since S is s*g-small in X_1, then there exists U as an s*g-neighborhood of y_1 which is thin relative to S. By (Mahmood, S. I. , 2015, proposition (1.7)), we have $U \times X_2$ is an s*g-neighborhood of (y_1, y_2) in $X_1 \times X_2$. Now, to prove that $((S,U)) = ((S \times X_2, U \times X_2))$.

$$g \in ((S,U)) \Leftrightarrow g.S \cap U \neq \phi \Leftrightarrow (g.S \cap U) \times X_2 \neq \phi \Leftrightarrow (g.S \times X_2) \cap (U \times X_2) \neq \phi$$
$$\Leftrightarrow (g.S \times g.X_2) \cap (U \times X_2) \neq \phi \Leftrightarrow g.(S \times X_2) \cap (U \times X_2) \neq \phi$$
$$\Leftrightarrow g \in ((S \times X_2, U \times X_2)).$$

Since $((S,U))$ is relatively compact in G, then so is $((S \times X_2, U \times X_2))$. Thus $S \times X_2$ is an s*g-small s*g-neighborhood of (x_1, x_2) in $X_1 \times X_2$, which means that $X_1 \times X_2$ is an s*g-proper G-space.

Theorem 2.8:

If $\psi : X_1 \to X_2$ is an equivariant homeomorphism function and X_1 is an s*g-proper G-space, then so is X_2.

Proof:

Since ψ is a homeomorphism function and X_1 is a completely regular Hausdorff space, then so is X_2. Let $y \in X_2$. Since ψ is an onto function, then there exists $x \in X_1$ such that $\psi(x) = y$. Since X_1 is an s*g-proper G-space, then x has an s*g-small s*g-neighborhood S in X_1. Since ψ is a homeomorphism, then by (Al-Meklafi, S., 2002, theorem (3.2.3)), we get $\psi(S)$ is an s*g-neighborhood of $\psi(x)$ in X_2. Now, to prove that $\psi(S)$ is s*g-small. Let $y' \in X_2$. Since ψ is an onto function, then there exists $x' \in X_1$ such that $\psi(x') = y'$. Since S is an s*g-small s*g-neighborhood of x in X_1 and $x' \in X_1$, then there exists U as an s*g-neighborhood of x' which is thin relative to S. By (Al-Meklafi, S., 2002, theorem (3.2.3)), we get $\psi(U)$ is an s*g-neighborhood of y' in X_2. Since ψ is an equivariant and 1-1 function, then we have:

$$g \in ((S,U)) \leftrightarrow g.S \cap U \neq \phi \leftrightarrow \psi(g.S \cap U) \neq \phi$$
$$\leftrightarrow \psi(g.S) \cap \psi(U) \neq \phi \leftrightarrow g.\psi(S) \cap \psi(U) \neq \phi$$
$$\leftrightarrow g \in ((\psi(S), \psi(U)))$$

Thus $((S,U)) = ((\psi(S), \psi(U)))$. But $((S,U))$ is relatively compact in G, then so is $((\psi(S), \psi(U)))$. That is $\psi(S)$ is an s*g-small s*g-neighborhood of y in X_2. Thus X_2 is an s*g-proper G-space.

Theorem 2.9:

If a G-space X has a star thin s*g-open set S, then X is an s*g-proper G-space.

Proof:

Let $x_1 \in X$. Since S is a star set, then $\exists\ g_1 \in G$ such that $g_1.x_1 \in S$. Therefore $x_1 \in g_1^{-1}.S$. Since $\phi_g : X \to X$ is a homeomorphism for each $g \in G$, then by (Al-Meklafi, S., 2002, theorem (3.2.3)), $g_1^{-1}.S$ is an s*g-neighborhood of x_1 in X. To prove that $g_1^{-1}.S$ is s*g-small. Let $x_2 \in X$, since S is a star set, then $\exists\ g_2 \in G$ such that $g_2.x_2 \in S$. Therefore $x_2 \in g_2^{-1}.S$ and $g_2^{-1}.S$ is an s*g-neighborhood of x_2 in X. But S is thin, then by (Palais, R. S. , 1961, definition (1.1.1)), we get $((g_1^{-1}.S, g_2^{-1}.S))$ is relatively compact in G. That is $g_1^{-1}.S$ is an s*g-small s*g-neighborhood of x_1 in X. Thus X is an s*g-proper G-space.

Theorem 2.10:

The topological sum of any two s*g-proper G-spaces is also s*g-proper G-space.

Proof:

Let X_1 and X_2 be any two s*g-proper G-spaces. By (Al-Jeburi, S. S., 2001, theorem (3.1.2)), $X_1 \cup X_2$ is a G-space. To prove that $X_1 \cup X_2$ is s*g-proper. Let $x \in X_1 \cup X_2$, then either $x \in X_1$ or $x \in X_2$. Suppose that $x \in X_1$. Since X_1 is an s*g-proper G-space, then there exists S as an s*g-small s*g-neighborhood of x in X_1. By proposition (1.12), S is an s*g-neighborhood of x in $X_1 \cup X_2$. Now, to prove that S is s*g-small in $X_1 \cup X_2$. For each $y \in X_1 \cup X_2$, then either $y \in X_1$ or $y \in X_2$. Hence we have two cases:

Case(1): Let $y \in X_1$. Since S is s*g-small in X_1, then there exists U as an s*g-neighborhood of y which is thin relative to S. By proposition (1.12), we get U is an s*g-neighborhood of y in $X_1 \bigcup X_2$ such that $((S,U))$ is relatively compact in G.
Case(2): Let $y \in X_2$. By proposition (1.12), we can take $U = X_2$ as an s*g-neighborhoods of y in $X_1 \bigcup X_2$, then we have: $((S, X_2)) = \{g \in G : g.S \bigcap X_2 \neq \phi\} = \phi$, since $GX_1 \subseteq X_1$ and $X_1 \bigcap X_2 = \phi$. Hence $((S, X_2))$ is relatively compact in G. That is in both cases, we get each point in $X_1 \bigcup X_2$ has an s*g-small s*g-neighborhood in $X_1 \bigcup X_2$. Thus $X_1 \bigcup X_2$ is an s*g-proper G-space.

3. Almost Semi Star Generalized Proper G-spaces

Definition 3.1:
A G-space X is called almost semi star generalized proper (almost s*g-proper) if there exists an s*g-open, dense, G-invariant, and connected subset A of X such that:
i) $X - A$ is 0-dimensional.
ii) A is an s*g-proper G-space.

Examples 3.2:
1) Every connected s*g-proper G-space X is an almost s*g-proper G-space, since X is an s*g-open, dense, G-invariant, and connected subset of X such that:
i) $X - X = \phi$ is 0-dimensional.
ii) X is an s*g-proper G-space.

2) Let X be any locally compact and connected s*g-proper G-space, then its one-point compactification X^* is an almost s*g-proper G-space.

Proof:
By (Al-Asadi, B.J., 2002), X^* is a G- space. To prove that X^* is almost s*g-proper.
Since $\overline{X} = X^*$, then X is an s*g-open, dense, G-invariant, and connected subset of X^* such that:
i) $X^* - X = \{x\}$ is 0-dimensional
ii) X is an s*g-proper G-space.

Theorem 3.3:
Let X be a connected s*g-proper G-space. Then the suspension SX of X is an almost s*g-proper G-space.

Proof:
By (Al-Attar, A.I. , 2000, lemma (2.1)), SX is a G-space. To show that SX is almost s*g-proper. Since X and the open interval $(0,1)$ are connected, then by (Mahmood, S. I. , 2015, proposition (1.7) and Willard, S., 1970), $A = X \times (0,1)$ is an s*g-open connected subset of SX.
Also for each $[x,k] \in A \Rightarrow k \neq 0$ and $k \neq 1$. Since $g.[x,k] = [g.x,k] = (g.x,k) \in X \times (0,1) = A \ \forall g \in G$. Hence A is G-invariant.
Clearly each of $[x,0]$ and $[x,1]$ is a limit point of A and hence $\overline{A} = SX$.
So $A = X \times (0,1)$ is an s*g-open, dense, G-invariant, and connected subset of SX.
Now,
i) Since $SX - A = \{[x,1],[x,0]\}$ is a discrete space $\forall x \in X$. Then $\{[x,1]\}$ and $\{[x,0]\}$ are clopen neighborhood bases at $[x,1]$ and $[x,0]$ respectively. Hence $SX - A$ is 0-dimensional.
ii) Let $[x,k] \in A \Rightarrow x \in X$. Since X is s*g-proper, then there exists an s*g-small s*g-neighborhood S of x in X. By (Mahmood, S. I. , 2015, proposition (1.7)), $S \times (0,1)$ is an s*g-neighborhood of $[x,k]$ in A. To show that $S \times (0,1)$ is s*g-small in A. Let $[x',k'] \in A \Rightarrow x' \in X$. Since S is s*g-small, then there exists an s*g-neighborhood U of x' which is thin relative to S. By (Mahmood, S. I. , 2015, proposition (1.7)), we have $U \times (0,1)$ is an s*g-neighborhood of $[x',k']$ in A.
Now, to show that $((S,U)) = ((S \times (0,1), U \times (0,1)))$.

$$g \in ((S,U)) \Leftrightarrow g.S \bigcap U \neq \phi \Leftrightarrow (g.S \bigcap U) \times (0,1) \neq \phi$$
$$\Leftrightarrow (g.S \times (0,1)) \bigcap (U \times (0,1)) \neq \phi$$

$\Leftrightarrow g \in ((S \times (0,1), U \times (0,1)))$.

Hence $((S,U)) = ((S \times (0,1), U \times (0,1)))$. Since $((S,U))$ is relatively compact in G, then so is $((S \times (0,1), U \times (0,1)))$. This means that $S \times (0,1)$ is an s*g-small s*g-neighborhood of $[x,k]$ in A. Since X is a completely regular Hausdorff space, then so is $A = X \times (0,1)$. Hence A is an s*g-proper G-space. Therefore SX is an almost s*g-proper G-space.

Remark 3.4:

An almost s*g-proper G-space may not be s*g-proper (not be proper) G-space.

Example 3.5:

If X is a connected s*g-proper G-space, then SX is an almost s*g-proper G-space, but is not s*g-proper (not proper) G-space.

Proof:

From theorem (3.3), SX is an almost s*g-proper G-space. To prove that SX is not s*g-proper (not proper) G-space. Assume that SX is an s*g-proper G-space $\Rightarrow \forall [x,k] \in SX$, $[x,k]$ has an s*g-small s*g-neighborhood S in SX. i.e. each $[x',k'] \in SX$ has an s*g-neighborhood U which is thin relative to S.

Take $[x,k] = [x',k'] = [x,0] \equiv X \times \{0\} \Rightarrow [x,0] \in S$ and $[x,0] \in U$. Hence for each $g \in G$, $g.[x,0] = [g.x,0] = X \times \{0\} = [x,0]$. Thus $g.[x,0] \in g.S \cap U$ for each $g \in G$. Therefore $g \in ((S,U))$ $\forall g \in G$. Which means that $((S,U)) = G$. Since $((S,U))$ is relatively compact in G, thus G is compact. This is a contradiction, since G is not compact. Hence SX is not an s*g-proper G-space. Since every proper G-space is s*g-proper G-space, then SX is not a proper G-space.

Remark 3.6:

A proper (s*g-proper) G-space may not be an almost s*g-proper G-space.

Example 3.7:

(Z,+) with the discrete topology is a locally compact non-compact topological group, also Z with the discrete topology is a completely regular Hausdorff space. Then Z acts on itself as follows: $n.m = n + m$ for each $n, m \in Z$. Clearly, Z is a proper Z-space, therefore Z is an s*g-proper Z-space. But Z is not almost s*g-proper Z-space, since the only dense subset of Z is Z itself, but Z is not connected.

Theorem 3.8:

If X is a connected G-space such that any dense proper subset of X is open star, then X is an s*g-proper G-space iff X is an almost s*g-proper G-space.

Proof:

$\Rightarrow$ It is obvious.

$\Leftarrow$ Since X is an almost s*g-proper G-space, then there exists a subset A of X which is an s*g-open, dense, G-invariant, and connected such that:

i) $X - A$ is 0-dimensional.

ii) A is an s*g-proper G-space.

If $A = X$, then X is a s*g-proper G-space.

If $A \neq X$, then A is an open star set.

To prove that X is an s*g-proper G-space.

Let $x \in X$. Since A is a star set, then there is $g_1 \in G$ such that $g_1.x \in A$. Since A is s*g-proper, then $g_1.x$ has an s*g-small s*g-neighborhood S in A. Since S is s*g-neighborhood in A and A is open in X, then by proposition (1.12), S is s*g-neighborhood in X. Since $g_1.x \in S$, then $x \in g_1^{-1}.S$. But $\phi_g : X \to X$ is a homeomorphism for each $g \in G$, then by (Al-Meklafi, S., 2002, theorem (3.2.3)), $g_1^{-1}.S$ is an s*g-neighborhood of x in X. To show that $g_1^{-1}.S$ is s*g-small in X. Let $y \in X$. Since A is a star set, then there is $g_2 \in G$ such that $g_2.y \in A$. Since S is s*g-small in A, then $g_2.y$ has an s*g-neighborhood U which is thin relative to S. As before $g_2^{-1}.U$ is an s*g-neighborhood of y in X. Since $((S,U))$ is relatively compact in G, then so is $((g_1^{-1}.S, g_2^{-1}.U))$. Thus X is an s*g-proper G-space.

Theorem 3.9:

If $\psi: X_1 \to X_2$ is an equivariant homeomorphism function and X_1 is an almost s*g-proper G-space, then so is X_2.

Proof:

Since ψ is a homeomorphism function and X_1 is a completely regular Hausdorff space, then so is X_2. Hence X_2 is a G-space. Now, to prove that X_2 is almost s*g-proper. Since X_1 is an almost s*g-proper G-space, then there exists a subset A of X_1 which is an s*g-open, dense, G-invariant and connected such that:

i) X-A is 0-dimensional.

ii) A is an s*g-proper G-space.

Since ψ is a homeomorphism function, then by (Willard, S., 1970 and Al-Meklafi, S., 2002, theorem (3.2.3)), $\psi(A)$ is a connected, dense, and s*g-open subset of X_2. Since ψ is an equivariant function and A is a G-invariant subset of X_1, then $g.\psi(A) = \psi(g.A) = \psi(A)$ for each $g \in G$. Thus $\psi(A)$ is a G-invariant subset of X_2. Therefore $\psi(A)$ is an s*g-open, dense, G-invariant, and connected subset of X_2.

Now,

i) $X_2 - \psi(A) = \psi(X_1 - A)$ is 0-dimensional, since 0-dimensional is a topological property.

ii) $\psi(A)$ is an s*g-proper G-space (by theorem (2.8).

Thus X_2 is an almost s*g-proper G-space.

Theorem 3.10:

If X_1 and X_2 are locally compact and almost s*g-proper G-spaces then $X_1 \times X_2$ is an almost s*g-proper G-space.

Proof:

If X_1 is an almost s*g-proper G-space, then there exists an s*g-open, dense, G-invariant, and connected subset A_1 of X_1 such that:

i) $X_1 - A_1$ is 0-dimensional.

ii) A_1 is an s*g-proper G-space.

Also, if X_2 is an almost s*g-proper G-space, then there exists an s*g-open, dense, G-invariant, and connected subset A_2 of X_2 such that:

iii) $X_2 - A_2$ is 0-dimensional.

iv) A_2 is an s*g-proper G-space.

Now,

$$X_1 \times X_2 = [(X_1 - A_1) \cup A_1] \times [(X_2 - A_2) \cup A_2].$$

$$= (X_1 - A_1) \times [(X_2 - A_2) \cup A_2] \cup A_1 \times [(X_2 - A_2) \cup A_2].$$

$$= (X_1 - A_1) \times (X_2 - A_2) \cup [(X_1 - A_1) \times A_2] \cup (A_1 \times X_2).$$

Since $(X_1 - A_1) \times (X_2 - A_2) \cap ([(X_1 - A_1) \times A_2] \cup (A_1 \times X_2)) = \phi$

$\Rightarrow$ $[(X_1 - A_1) \times (X_2 - A_2)]^c = [(X_1 - A_1) \times A_2] \cup (A_1 \times X_2)$.

Since $(X - A)$ and $(Y - B)$ are s*g-closed in X_1 and X_2 respectively, then by (Mahmood, S. I. , 2015, proposition (1.7)), $(X_1 - A_1) \times (X_2 - A_2)$ is an s*g-closed set in $X_1 \times X_2$.Thus $[(X_1 - A_1) \times A_2] \cup (A_1 \times X_2)$ is s*g-open in $X_1 \times X_2$.

To show that $[(X_1 - A_1) \times A_2] \cup (A_1 \times X_2)$ is a dense subset of $X_1 \times X_2$. If not, then by (Willard, S., 1970), there is a non-empty open subset V of $X_1 \times X_2$, hence $V = V_1 \times V_2$, where V_1 and V_2 are non-empty open subset of X_1 and X_2 respectively and $(V_1 \times V_2) \cap ([(X_1 - A_1) \times A_2] \cup (A_1 \times X_2)) = \phi$.

$\Rightarrow ((V_1 \times V_2) \cap [(X_1 - A_1) \times A_2]) \cup ((V_1 \times V_2) \cap (A_1 \times X_2)) = \phi$.

Therefore $(V_1 \times V_2) \cap [(X_1 - A_1) \times A_2] = \phi$ and $(V_1 \times V_2) \cap (A_1 \times X_2) = \phi$.

$\Rightarrow V_1 \cap (X_1 - A_1) = \phi$ or $V_2 \cap A_2 = \phi$ and $V_1 \cap A_1 = \phi$ or $V_2 \cap X_2 = \phi$.

This is a contradiction, since $V_2 \subset X_2$ and $V_1 \cap A_1 \neq \phi$, because A_1 is dense in X_1.

Thus $[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is a dense subset of $X_1 \times X_2$.

Since X_1 and X_2 are G-Spaces, then G acts on X_1 and X_2 by ϕ_1 and ϕ_2 respectively.

Define $\phi : G \times X_1 \times X_2 \to X_1 \times X_2$ as follows:

$\phi(g,(x_1,x_2)) = (\phi_1(g,x_1),\phi_2(g,x_2)) = (g.x_1, g.x_2)$ for each $g \in G$ and $(x_1,x_2)\in X_1 \times X_2$.

By (Al-Jeburi, S. S., 2001), $X_1 \times X_2$ is a G-space.

Now, for each $g \in G$ and each $a = (a_1,a_2)\in (X_1 - A_1)\times(X_2 - A_2)$.

To prove that $\phi(g,a)\in (X_1 - A_1)\times(X_1 - A_2)$.

Since $a \in (X_1 - A_1)\times(X_2 - A_2) \Rightarrow a_1 \in (X_1 - A_1)$ and $a_2 \in (X_2 - A_2)$.

Since A_1 and A_2 are G-invariant subset of X_1 and X_2 respectively, then $(X_1 - A_1)$ and $(X_2 - A_2)$ are also G-invariant subset of X_1 and X_2 respectively.

$\Rightarrow \phi_1(g,a_1)\in (X_1 - A_1)$ and $\phi_2(g,a_2)\in (X_2 - A_2)$

$\Rightarrow (\phi_1(g,a_1),\phi_2(g,a_2))\in (X_1 - A_1)\times(X_2 - A_2)$

$\Rightarrow \phi(g,a)\in (X_1 - A_1)\times(X_2 - A_2)$.

$\Rightarrow$ $(X_1 - A_1)\times(X_2 - A_2)$ is a G-invariant subset of $X_1 \times X_2$. Since $[(X_1 - A_1)\times(X_2 - A_2)]^c =$
$[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2) \Rightarrow [(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is a G-invariant subset of $X_1 \times X_2$.

To show that $[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is a connected subset of $X_1 \times X_2$.

$A_1 \times A_2 \subseteq ([X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2) \subseteq X_1 \times X_2 = \overline{A_1}\times\overline{A_2} = \overline{A_1 \times A_2}$.

Since A_1 and A_2 are connected in X_1 and X_2 respectively, then by (Willard, S., 1970), $A_1 \times A_2$ and $\overline{A_1 \times A_2}$ are connected in $X_1 \times X_2$. Therefore $[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is a connected subset of $X_1 \times X_2$ (see Willard, S., 1970). So $[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is an s*g-open, dense, G-invariant, and connected subset of $X_1 \times X_2$.

Now,

i) Since $X_1 - A_2$ and $X_2 - A_2$ are s*g-closed sets in X_1 and X_2 respectively, therefore $X_1 - A_2$ and $X_2 - A_2$ are g-closed sets in X_1 and X_2 respectively. Since X_1 and X_2 are locally compact Hausdoff space, then by (Levin, N., 1970 and Willard, S., 1970), $X_1 - A_2$ and $X_2 - A_2$ are locally compact Hausdorff spaces. Since $X_1 - A_2$ and $X_2 - A_2$ are 0-dimensional, then by (Willard, S., 1970, theorem (29.7) and theorem (29.3)), $(X_1 - A_1)\times(X_2 - A_2)$ is 0-dimensional.

ii) Since A_1 and A_2 are s*g-proper, then by theorem (2.7) $(A_1 \times X_2)$ and $(X_1 - A_1)\times A_2$ are s*g-proper. Since $[(X_1 - A_1)\times A_2]\bigcap(A_1 \times X_2) = \phi$, then by theorem (2.10), $[(X_1 - A_1)\times A_2]\bigcup(A_1 \times X_2)$ is s*g-proper.

Thus $X_1 \times X_2$ is an almost s*g-proper G-space.

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