Australian Journal of Basic and Applied Sciences, 10(1) January 2016, Pages: 666-669

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

Quantum Mechanical Motion of an Electron In Inhomogeneous Electric and Magnetic Fields

S. Benjamin, G. Pandiaraj, C. Raja Rajeswari and S. Revathi

Department of Physics, VSB College of Engineering Technical Campus, Coimbatore, India

Address For Correspondence:
S. Benjamin, Department of Physics, VSB College of Engineering Technical Campus, Coimbatore, India
E-mail: sbenjamin@vsbcetc.com

ARTICLE INFO

Article history:
Received 12 February 2016
Accepted 12 March 2016
Available online 20 March 2016

Keywords:
Bound state solutions, Dirac equation, inhomogeneous fields, Kratzer's molecular, Laguerre polynomials.

ABSTRACT

Dirac equation governs the behavior of relativistic motion of spin ½ particles. Several instances of solution of Dirac equation in various field configurations can be seen in the literature. In this paper we present certain exact solutions of the Dirac equation for an electron in crossed inhomogeneous electric and magnetic fields. Closed form bound state solutions are obtained in terms of Laguerre polynomials. The eigenvalues reduce to the results in the literature in the limit of vanishing inhomogeneity and electric field.

INTRODUCTION

Dirac equation, (Dirac, P.A., 1928; Dirac, P.A.M., 1928) which describes the quantum and relativistic behavior of a spin ½ particle is one of the greatest achievements of 20^{th} century science. It is attributed to Dirac himself that the relativistic wave equation of the electron is the basis of all Chemistry and almost all of Physics (Rajasekaran, G., 2003).

$$\left[\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}+\frac{mc}{h}\right]\psi=0\ (x^{\mu}=(\boldsymbol{x},ict))$$

This equation is linear in $\frac{\partial}{\partial t}$ and it assures positive definite probability density which were lacking in Klein-Gordon equation. Notwithstanding the merits of the Dirac equation exact solutions of the equation have been obtained only for some configurations. Some of them are: Coulomb potential (Feynman, R.P., M. Gell-Mann, 1958), a constant magnetic field (Rabi, I.V., 1928), a constant electric field (Sauter, F., 1931), the field of a plane wave (Volkov, D.M., 1935), the field of a plane wave with a constant magnetic field parallel to the direction of propagation of plane wave (Redmond, P.J., 1965), four cases in which the electromagnetic potentials assume functional dependence on the space coordinates (Stanciu, G.N., 1966) and one where electric and magnetic fields are crossed (Lam, L., 1970). Two component form of the Dirac equation was solved in (Kulkarni, S.V., L.K. Sharma, 1979) by identifying it with the Schrodinger equation with the Kratzer's molecular potential. Preliminary work leading to certain exact solutions of the Dirac equation for the following four field configurations were worked out in [12]:

$$H_z=\frac{H}{(1-ay)^2},H_x=H_y=0 \quad (1)$$

$$H_z=\frac{H}{(1-ay)^2}\ ,H_x=\ H_y=0$$

$$E_y=\frac{E}{(1-ay)^2}\ ,E_x=E_z=0 \quad (2)$$

To Cite This Article: S. Benjamin, G. Pandiaraj, C. Raja Rajeswari and S. Revathi., Quantum Mechanical Motion of an Electron In Inhomogeneous Electric and Magnetic Fields. ***Aust. J. Basic & Appl. Sci.,*** *10(1): 666-669, 2016*

$H_z = \frac{H}{(1-ay)^2}$, $H_x = H_y = 0$
$E_z = \frac{E}{(1-az)^2}$, $E_x = E_y = 0$ (3)
$H_z = \frac{H}{(1-ay)^2}$, $H_x = H_y = 0$
$E_z = E,\ E_x = E_y = 0$ (4)

Solution for the following field was reported in [13]

$E_z = \frac{E}{(1-az)^2}, E_x = E_y = 0$ (5)

For the field (1) both bound state as well as scattering solutions are obtained (Canuto, V., C. Chiuderi, 1969). In respect of the two fields (2) and (3) only bound state solutions are presented. For the fields (4) and (5) the scattering solutions alone are given; no bound state solutions are possible for these case. In this paper we present the bound state solutions for field (2). The combined fields in (2) can be obtained by Lorentz transformation from a rest frame of reference where there exists only the field (1) to a frame moving with velocity E/H along the x-axis. Thus one can obtain the solution for the present case by Lorentz transformation of the solutions obtained for field (1). However we choose to solve the problem ab initio. It is worth mentioning here that the solutions of the Dirac equation for crossed homogeneous electric and magnetic fields have been obtained by straight forward procedure by and by a Lorentz transformation of the homogeneous field solutions by.

1.1. The Dirac Equation:

It is convenient to start with the two-component form of the Dirac equation (DE).

$$[(\boldsymbol{p} + e\boldsymbol{A})^2 + m^2 + e\boldsymbol{\sigma}.(\boldsymbol{H} + i\boldsymbol{E})]\psi = (W + eV)^2\,\psi \tag{6}$$

The σ's are the Pauli spin matrices and we use the natural units ħ = c = 1. The four-component Dirac spinor ψ_D which is the solution of the DE

$$[\gamma_j(\boldsymbol{p} + e\boldsymbol{A}) + i\gamma_4(W + eV) - im]\psi_D = 0 \tag{7}$$

is generated from the solutions of the two component equation (5) as follows. We write

$$\psi_D = \begin{bmatrix} [\boldsymbol{\sigma}.(\boldsymbol{p} + e\boldsymbol{A}) + W + eV + m]\psi \\ [\boldsymbol{\sigma}.(\boldsymbol{p} + e\boldsymbol{A}) + W + eV - m]\psi \end{bmatrix} \tag{8}$$

The fields (2) considered in this work are derived from the gauge:

$$A_x(y) = -\frac{Hy}{(1-ay)},\ A_y = 0, A_z = 0$$
$$V(y) = -\frac{Ey}{(1-ay)} \tag{9}$$

1.2. Bound state solutions for the Field Configuration (2):

For magnetic field under consideration, the two component DE can be written as

$$\left\{p_y^2 + p_z^2 + \left[p_x - \frac{eHy}{(1-ay)}\right]^2 + m^2 + \frac{e\Lambda}{(1-ay)^2}\right\}\psi = \left[W - \frac{eEy}{1-ay}\right]^2\psi \tag{10}$$
$$\Lambda = \begin{bmatrix} H & E \\ -E & -H \end{bmatrix}$$

The spin part of (9) can be diagonalised with the matrix

$$S = \begin{bmatrix} \Gamma_- & \Gamma_+ \\ -\Gamma_+ & -\Gamma_- \end{bmatrix}, \Gamma_\pm^2 = \tfrac{1}{2}(\gamma \pm 1), \gamma^{-2} = 1 - \beta^2, \beta = E/H \tag{11}$$

It is to be noted that the diagonalization of the spin part of (10) is identical to the one in the homogeneous case. Thus we have

$$[Q - seH\gamma^{-1}/(1-ay)^2]X_s = 0 \tag{12}$$

Where

$$X_s = S\psi_s \tag{13}$$
$$Q = p_x^2 + p_y^2 + p_z^2 + \xi^2\gamma^{-2} + 2\xi(p_x - \beta W) + m^2 - W^2 - \frac{2\xi(p_x - \beta W)}{1-ay} - \frac{2\xi^2\gamma^{-2}}{1-ay} + \frac{\xi^2\gamma^{-2}}{(1-ay)^2} \tag{14}$$

Introducing a new variable

$$\eta = \frac{2}{a}k(1-ay)$$

with

$$k = [-W^2 + p_x^2 + p_z^2 + \xi^2\gamma^{-2} + m^2 + 2\xi(p_x - \beta W)]^{1/2} \tag{15}$$

The equation (11) reduces to

$$\left\{\frac{d^2}{d\eta^2} - \frac{1}{4} + \frac{\xi(p_x - \beta W) + \xi^2\gamma^{-2}}{ka\eta} - \frac{\xi^2\gamma^{-2} + esH\gamma^{-1}}{a^2\eta^2}\right\}X_s(\eta) = 0 \tag{16}$$

This is in the form of the well-know Whittaker equation (18)

$$\left\{\frac{d^2}{d\eta^2} - \frac{1}{4} + \frac{\chi}{\eta} + \frac{\frac{1}{4}-\mu^2}{\eta^2}\right\} X_s(\eta) = 0 \qquad (17)$$

The parameters χ and μ in (17) are identified as

$$\chi = \frac{\xi(p_x - \beta W) + \xi^2\gamma^{-2}}{ka} \qquad (18)$$

$$\mu = \frac{\xi}{\gamma} + \frac{1}{2}s \qquad (19)$$

Now the solution of (16) for $\eta > 0$ is obtained as

$$X_s(\eta) = e^{-\eta/2}\eta^{\mu+1/2} 1F_1\left(\mu + \frac{1}{2} - \chi; 1 + 2\mu\, ;\, \eta\right) \qquad (20)$$

Where $_1F_1$ ( ) is the confluent hypergeometric function (also known as Kummer function). This function has the following asymptotic values:

$$_1F_1(\mu + ½ -\chi;\, 1 + 2\mu\, ;\, \eta) \;\; \eta \overset{\approx}{\rightarrow} \infty\; e^{\eta}\eta^{-\mu-\frac{1}{2}-\chi} \qquad (21)$$

However, if

$$\mu + \frac{1}{2} - \chi = -n, n = 0, 1, 2 \,.... \qquad (22)$$

then $_1F_1$ (-n; 1 + 2μ; η) reduces to the generalized Laguerre polynomial $L_n^{(2\mu)}(\eta)$. Thus we get the solution $X_s(\eta)$ which has the proper asymptotic behavior at $\eta = \infty$ as

$$X_s(\eta) = C_n e^{-\frac{|\eta|}{2}}|\eta|^{\mu+\frac{1}{2}} L_n^{(2\mu)}(\eta) \qquad (23)$$

where C_n is the normalization constant and $L_n^{(2\mu)}(\eta)$ is the generalized Laguerre polynomial.

The energy eigenvalues are given by

$$W_{n,s} = -eH\beta[aN(a^2N + 2eH\gamma^{-1}) - eHp_x] \pm (eH\beta)^2[a^2N(a^2N + 2eH\gamma^{-1}) - eHp_x]^2 + [(a^2N + eH\gamma^{-1})^2 + (eH\beta)^2][(a^2N + eH\gamma^{-1})^2(m^2 + p_z^2 - (\beta\gamma p_x)^2 + (a\gamma p_x + eH\gamma^{-1})^2(a^2N^2 + 2NeH\gamma^{-1})]^{1/2} / [(a^2N + eH\gamma^{-1})^2 + (eH\beta)^2] \qquad (24)$$

It can be seen from the above expression that when β = 0 (i.e., no electric field) $W_{n,s}$ reduces to the one in field (1).

$$W_{n,s}^2 = m^2 + p_z^2 + (ap_x + eH)^2 N \frac{(a^2N + eH)}{(a^2N + eH)^2} \qquad (25)$$

Further, when a= 0 we get from (24)

$$W_{n,s} = \beta p_x \pm \gamma^{-1}[m^2 + p_z^2 + 2NeH\gamma^{-1}]^{1/2} \qquad (26)$$

which is the energy of the electron in crossed homogeneous electric and magnetic fields.

II. Conclusion:

The problem of particles and fields has been one of the exciting areas of research holding the attention of both theoreticians and experimentalists. The results would be of interests to researchers in the fields of astrophysics, plasma physics, condensed matter physics and particle physics among others. In this paper bound state solutions of Dirac equation in inhomogeneous electric and magnetic field configuration is presented. The transition in terms of the eigenvalues to (i) inhomogeneous magnetic field and (ii) Crossed homogeneous electric and magnetic fields is indicated. The fact that these fields give rise to exact solutions for the classical as well as quantum mechanical motion, amidst paucity of such rigorous solutions encourages one to study the electron behavior in these fields. Using the eigenvalues and proper computation of density of state and degeneracy one can compute statistical properties of electron gas in the external field. However, complete effect of the inhomogeneous fields will be known on studying quantum electrodynamic processes which, of course make use of the eigenfunctions as well. It may be worth pondering why classical equations such as Laplace equation and wave equation have rich solutions while the solutions of Dirac equations are just handful.

ACKNOWLEDGMENT

The authors would like to thank Shri. V. Balsamy, Chairman and B. Satheeshkumar, Secretary, of VSB Educational Trust for providing the necessary facility for preparing this paper. The encouragement given by Dr. P. Venugopal, Principal, VSB College of Engineering Technical Campus is acknowledged. One of the authors (S. B) thanks Dr. P. Achuthan, Retired Professor, IIT Madras and Dr. K. Venkatesan for their guidance during the preliminary work reported in this paper.

REFERENCES

Dirac, P.A., 1928. The quantum theory of the electron. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences,* 117(778): 610-624. The Royal Society.

Dirac, P.A.M., 1928. The quantum theory of the electron. Part II. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences,* 118(779): 351-361. The Royal Society.

Rajasekaran, G., 2003. The discovery of Dirac equation and its impact on present-day physics. *Resonance*, *8*(8): 59-74.

Feynman, R.P., M. Gell-Mann, 1958. Theory of the Fermi interaction. *Physical Review*, *109*(1): 193.

Rabi, I.V., 1928. Das freie Elektron im homogenen Magnetfeld nach der Diracschen Theorie. *Zeitschrift für Physik A Hadrons and Nuclei*, *49*(7): 507-511.

Sauter, F., 1931. Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs. *Zeitschrift für Physik*, *69*(11-12): 742-764.

Volkov, D.M., 1935. On a class of solutions of the Dirac equation. *Z. Phys*, *94:* 250.

Redmond, P.J., 1965. Solution of the Klein-Gordon and Dirac Equations for a Particle with a Plane Electromagnetic Wave and a Parallel Magnetic Field. *Journal of Mathematical Physics*, *6*(7): 1163-1169.

Stanciu, G.N., 1966. Solvable vector and scalar potentials for the Dirac equation. *Physics Letters*, *23*(3): 232-233.

Lam, L., 1970. New exact solutions of the Dirac equation. *Canadian Journal of Physics*, *48*(16): 1935-1937.

Kulkarni, S.V., L.K. Sharma, 1979. Exact solutions of Dirac equation. *Pramana*, *12*(5): 475-480.

Gordon, W., 1928. Die Energieniveaus des Wasserstoffatoms nach der Diracschen Quantentheorie des Elektrons. *Zeitschrift für Physik A Hadrons and Nuclei*, *48*(1): 11-14.

Hadjikonstantinou, P., A. Jannussis, 1975. Relativistic motion of an electron in uniform external electric and magnetic fields. *Fizika: A Journal of Experimental and Theoretical Physics*, *7:* 177-186.

Canuto, V., C. Chiuderi, 1969. Solution of the Dirac equation in orthogonal electric and magnetic fields. *Lettere Al Nuovo Cimento (1969-1970)*, *2*(6): 223-227.