Australian Journal of Basic and Applied Sciences, 10(12) July 2016, Pages: 267-274

AUSTRALIAN JOURNAL OF BASIC AND APPLIED SCIENCES

ISSN:1991-8178 EISSN: 2309-8414
Journal home page: www.ajbasweb.com

Quadratic Finite Element Solution to the Coupled-BBM System

[1]Muhannad A.M. Shallal and [2]Mohammed M. Rasheed

[1]*Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.*
[2]*Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.*

Address For Correspondence:
Muhannad A.M. Shallal, Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq
E-mail: Muhannad_ker@yahoo.com

ARTICLE INFO

Article history:
Received 28 May 2016
Accepted 29 July 2016
Published 17 August 2016

Keywords:
Coupled BBM-System, Finite Element Method, Solitary Waves.

ABSTRACT

In this paper, numerical solution of the coupled BBM-system of Boussinesq type based finite element method is proposed. Quadratic element has been used in this paper. on The performance of the method is illustrated by studying a solitary wave motion and the interaction of two solitary waves. The results are compared against analytic solution of the system to demonstrate the efficiency of the proposed method.

INTRODUCTION

We consider the Coupled-BBM system, which belongs to the class of Boussinesq systems, modeling two-way propagation of long waves of small amplitude on the surface of water in a channel**.** The system is a good candidate for modeling long waves of small to moderate amplitude. The Coupled BBM-system is given by (Bona and Chen, 1998).

$$\begin{aligned} &v_t + u_x + (vu)_x - \frac{1}{6} v_{xxt} = 0 \\ &u_t + v_x + uu_x - \frac{1}{6} u_{xxt} = 0 \end{aligned} \qquad (1)$$

where x corresponds to distance along the channel and t is the elapsed time, $v(x,t)$ is a dimensionless deviation of the water surface from its undisturbed position and $u(x,t)$ is the dimensionless horizontal velocity above the bottom of the channel.

The theoretical results like existence of line solitary waves and the satiability of the solitary wave solution, have been discussed in (Bona and Chen, 1998) and (Chen and Wang, 2015). We refer the reader to (Chen *et al.*, 2007) who derived the existence of periodic traveling-wave solutions $(v(x,t), u(x,t))$ of the form

$$v(x,t) = v(x-wt) = \sum_{n=-\infty}^{\infty} v_n e^{i(n\pi/l)(x-wt)}$$

$$u(x,t) = u(x-wt) = \sum_{n=-\infty}^{\infty} u_n e^{i(n\pi/l)(x-wt)}$$

To Cite This Article: Muhannad A.M. Shallal and Mohammed M. Rasheed., Quadratic Finite Element Solution to the Coupled-BBM System. ***Aust. J. Basic & Appl. Sci.,*** *10(12): 267-274, 2016*

Where l and w denote the half-period and the phase speed, respectively.

Rigorous errors estimate for Coupled-BBM type systems was proved in (Chatzipantelidis, 1998). One of the advantages that (1) has over alternative Boussinesq-type systems is the easiness with which it may be integrated numerically (Bona *et al*., 2005). Furthermore, it was proved in (Chen, 2000), (Bona *et al*., 2005) and (Bona *et al*., 2004) that the initial value problem either for $x \in \Re$ or with boundary conditions ($x \in [a,b]$) for (1) is well posed in certain natural function classes.

The initial-boundary value problem of the form (1) posed on a bounded smooth plane domain with homogenous Dirichlet or Neumann or reflective (mixed) boundary conditions which is locally well-posed (Dougalis *et al*., 2009).

(Mahmoud, 2013) solved the Coupled BBM-system using finite element method adopting linear element in the case of Dirichlet boundary condition. In this paper the Coupled BBM-system will be solved using quadratic element corresponding to the Dirichlet and Neumann boundary conditions.

The structure of the paper is as follows. In section 2, quadratic finite element method is designed for the numerical solution of the Coupled BBM-system. The results and discussions are presented in section 3. In the last section, a summary of the main conclusions is given at the end of the paper.

2. Finite Element Solution to the Coupled-BBM system:

Consider the system (1) and let u^j and v^j represent the discretized solution for this system. Then system (1) can be expressed as:

$$v_t^j + u_x^j + (v^j u^j)_x - \frac{1}{6} v_{xxt}^j = 0 \qquad (2a)$$

$$u_t^j + v_x^j + u^j u_x^j - \frac{1}{6} u_{xxt}^j = 0 \qquad (2b)$$

In this study, we consider two different boundary conditions (Dirichlet and Neumann). Multiply equation (2a) by a test function $w \in W(\Omega)$ where $\Omega = (a,b), \quad a,b \in \Re$ and integrate over the finite element (x_e, x_{e+1}) with the length h, we get the following equation

$$\int_{x_e}^{x_{e+1}} (wv_t^j + wu_x^j + wv^j u_x^j + wu^j v_x^j - \frac{1}{6} wv_{xxt}^j) dx = 0$$

And this gives

$$\int_{x_e}^{x_{e+1}} (wv_t^j + wu_x^j + wv^j u_x^j + wu^j v_x^j + \frac{1}{6} w_x v_{xt}^j) dx = \frac{1}{6} wv_{xt} \Big|_{x_e}^{x_{e+1}} \qquad (3)$$

where the right-hand side of equation (3) is evaluated only at the boundaries, and it will be zero when we using Dirichlet or Neumann boundary conditions.

The approximate solution can be written:

$$v(x) = \sum_{s=1}^{n_e} v_s(t) N_s(x)$$

$$u(x) = \sum_{s=1}^{n_e} u_s(t) N_s(x) \qquad (4)$$

$$w(x) = N_i(x), \qquad i = 1,...,n_e$$

Where $u_s^j(t)$ and $v_s^j(t), s = 1,...,n_e$ are unknowns and $N_s(x)$ are the quadratic basis function. Then

$$\sum_{s=1}^{n_e} \int_0^h (N_i(x) N_s(x) dx + \frac{1}{6} \frac{dN_i(x)}{dx} \frac{dN_s(x)}{dx} dx) \overset{\bullet}{v}{}_s^j + \sum_{s=1}^{n_e} N_i(x) \frac{dN_s(x)}{dx} u_s^j dx$$

$$+ \int_0^h N_i(x) \sum_{s=1}^{n_e} N_s(x) u_s^j \sum_{s=1}^{n_e} \frac{dN_s(x)}{dx} v_s^j dx + \int_0^h N_i(x) \sum_{s=1}^{n_e} N_s(x) v_s^j \sum_{s=1}^{n_e} \frac{dN_s(x)}{dx} u_s^j dx = 0 \qquad (5)$$

Where $\overset{\bullet}{v}{}_s^j \approx \dfrac{v_s^{j+1} - v_s^j}{\Delta t}$, equation (5) can be simplified to the following matrix equation

$$[A^e + B^e]\{v^{j+1}\} = [A^e + B^e]\{v^j\} - \Delta t F^e(u^j, v^j) \quad (6)$$

Where

$$A_{i,s}{}^k = \int_0^h N_i(x)N_s(x)dx = \begin{bmatrix} \int_{e_k} N_{k-1}(x)N_{k-1}(x)dx & \int_{e_k} N_{k-1}(x)N_{k-1/2}(x)dx & \int_{e_k} N_{k-1}(x)N_k(x)dx \\ \int_{e_k} N_{k-1/2}(x)N_{k-1}(x)dx & \int_{e_k} N_{k-1/2}(x)N_{k-1/2}(x)dx & \int_{e_k} N_{k-1/2}(x)N_k(x)dx \\ \int_{e_k} N_k(x)N_{k-1}(x)dx & \int_{e_k} N_k(x)N_{k-1/2}(x)dx & \int_{e_k} N_k(x)N_{k-1}(x)dx \end{bmatrix}$$

$$B_{i,s}{}^k = \frac{1}{6}\int_0^h \frac{dN_i(x)}{dx}\frac{dN_s(x)}{dx}dx = \frac{1}{6}\begin{bmatrix} \int_{e_k} \frac{dN_{k-1}}{dx}\frac{dN_{k-1}}{dx}dx & \int_{e_k} \frac{dN_{k-1}}{dx}\frac{dN_{k-1/2}}{dx}dx & \int_{e_k} \frac{dN_{k-1}}{dx}\frac{dN_k}{dx}dx \\ \int_{e_k} \frac{dN_{k-1/2}}{dx}\frac{dN_{k-1}}{dx}dx & \int_{e_k} \frac{dN_{k-1/2}}{dx}\frac{dN_{k-1/2}}{dx}dx & \int_{e_k} \frac{dN_{k-1/2}}{dx}\frac{dN_k}{dx}dx \\ \int_{e_k} \frac{dN_k}{dx}\frac{dN_{k-1}}{dx}dx & \int_{e_k} \frac{dN_k}{dx}\frac{dN_{k-1/2}}{dx}dx & \int_{e_k} \frac{dN_k}{dx}\frac{dN_k}{dx}dx \end{bmatrix} \quad (7)$$

and

$$\{F^e(u^j, v^j)\}_i = \int_0^h \sum_{s=1}^{n_e} N_i(x)\frac{dN_s(x)}{dx}u_s^j dx + \int_0^h N_i(x)\sum_{s=1}^{n_e} N_s(x)u_s^j \sum_{s=1}^{n_e}\frac{dN_s(x)}{dx}v_s^j dx + \int_0^h N_i(x)\sum_{s=1}^{n_e} N_s(x)v_s^j \sum_{s=1}^{n_e}\frac{dN_s(x)}{dx}u_s^j dx \quad (8)$$

By the same way, equation (2b) becomes:

$$\sum_{s=1}^{n_e}\int_0^h (N_i(x)N_s(x)dx + \frac{1}{6}\frac{dN_i(x)}{dx}\frac{dN_s(x)}{dx}dx)\overset{\bullet}{u}{}_s^j + \sum_{s=1}^{n_e} N_i(x)\frac{dN_s(x)}{dx}v_s^j dx + \int_0^h N_i(x)\sum_{s=1}^{n_e} N_s(x)u_s^j \sum_{s=1}^{n_e}\frac{dN_s(x)}{dx}u_s^j dx = 0 \quad (9)$$

Where $\overset{\bullet}{u}{}_s^j \approx \frac{u_s^{j+1} - u_s^j}{\Delta t}$. This will also simplify to the following matrix equation

$$[A^e + B^e]\{u^{j+1}\} = [A^e + B^e]\{u^j\} - \Delta t G^e(u^j, v^j) \quad (10)$$

where A^e, B^e as in equation (7), and

$$\{G^e(u^j, v^j)\}_i = \int_0^h \sum_{s=1}^{n_e} N_i(x)\frac{dN_s(x)}{dx}v_s^j dx + \int_0^h N_i(x)\sum_{s=1}^{n_e} N_s(x)u_s^j \sum_{s=1}^{n_e}\frac{dN_s(x)}{dx}u_s^j dx$$

It is clear that for the quadratic elements, A^e and B^e are the following matrices

$$A^k = \frac{h}{30}\begin{pmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{pmatrix}, \quad B^k = \frac{1}{18h}\begin{pmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{pmatrix}$$

Now we divide the region [a,b] into N finite elements of equal length h where $h = \frac{(b-a)}{N}$ by knots x_i such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. We get the following global matrix equation of $(n-1)$ by $(n-1)$ for our system after assembling the local element equations (6) and (10)

$$\begin{aligned} [A+B]\{v^{j+1}\} &= [A+B]\{v^j\} - \Delta t F(u^j, v^j), \\ [A+B]\{u^{j+1}\} &= [A+B]\{u^j\} - \Delta t G(u^j, v^j) \end{aligned} \quad (11)$$

3. Numerical results:

Quadratic finite element solution of the coupled-BBM system with the initial condition and two types of boundary conditions have been presented in this section.

To illustrate the efficiency of the method, we compute L_2 error.

$$L_2 = \left\| u^{exact} - u^{num} \right\|_2 = \sqrt{h \sum_{j=0}^{N} \left| u_j^{exact} - u_j^{num} \right|^2}$$

Single Solitary Wave Motion:

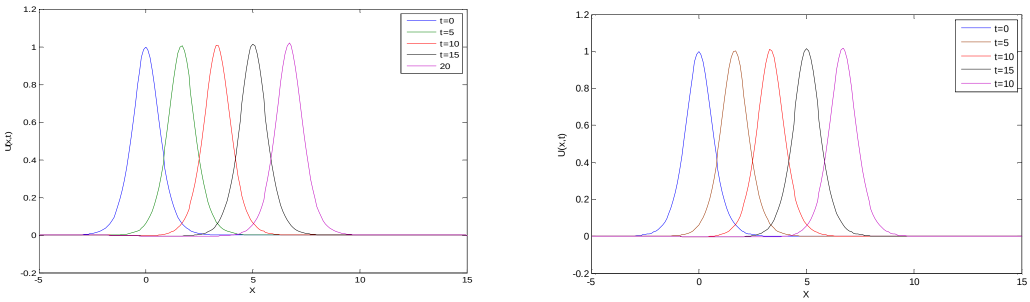
The motion of solitary waves is considered in this section. It is well known that system (1) posses analytical solution of the form (Chen,1998).

$$v(x,t) = -1$$

$$u(x,t) = (1 - \frac{g}{6})c + \frac{cg}{2} \operatorname{sech}^2 (\frac{\sqrt{g}}{2}(x + x_0 - ct)) \tag{12}$$

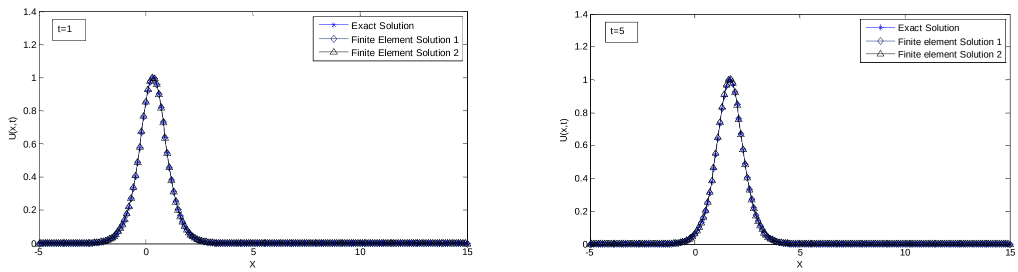
where g, x_0 and c are real constants. To compare our results with (12), equation (12) is taken as the initial condition and all computations in the simulations, assume $x_0 = 0, g = 6$ and $c = \frac{1}{3}$, for $-5 \le x \le 15$ so that the solitary wave has an amplitude of 1.

Our simulations have been executed up to time t=20. In table 1, we show the L_2 error at $\Delta t = 0.01,$ with various time and step sizes. In figure1, single solitary wave profiles are graphed at different times t=1, 5, 10, 15, and t=20 with $\Delta t = 0.01, \Delta x = 0.1, -5 \le x \le 15$ for the two types of boundary conditions. Both the analytical and numerical solutions are graphed at different time shown in figure 2 and the plots of those solutions are indistinguishable. The error distributions of the Quadratic finite element method and analytic solution at t=1, 10 and t=20 are plotted in figure3.



Neumann B. C   Dirichlet B. C

Fig. 1: Single solitary wave profiles with $\Delta t = 0.01, \Delta x = 0.1, -5 \le x \le 15$ at level times; $t = 0,5,10,15$ and 20.



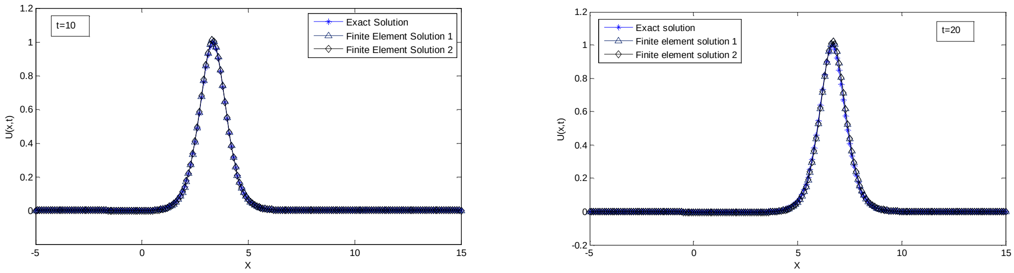


Fig. 2: Comparison between analytic solution, Quadratic finite element solution with Dirichlet boundary condition and Quadratic finite element solution with Neumann boundary condition at t=1, 5, 10 and t=20.

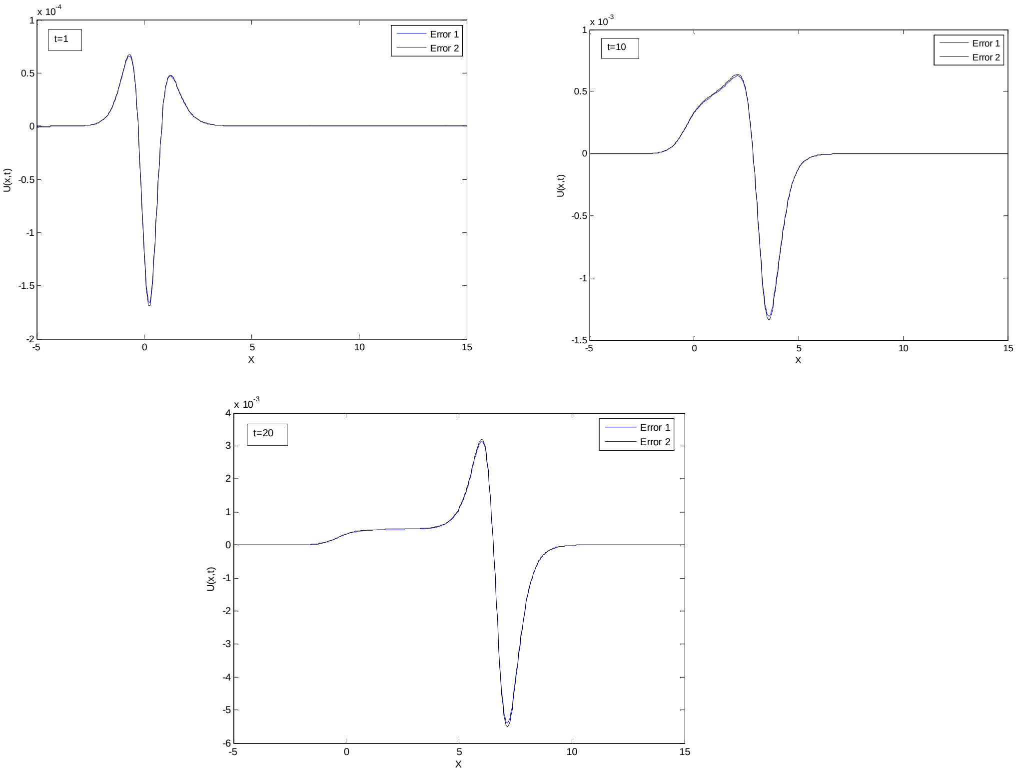


Fig. 3: Graphs of errors at t=1, 10 and t=20 (note: error 1 and error 2, refers to the errors using Dirichlet and Neumann boundary conditions, respectively)

Table 1: The error norm at different time and step sizes when $\Delta t = 0.01,\ -5 \le x \le 15$.

T	Δx	L_2 (Neu.BC)	L_2 (Dir. BC)	Δx	L_2 (Neu.BC)	L_2 (Dir. BC)
1	0.1	3.6855E-5	4.6587E-5	0.05	2.9739E-5	3.4421E-5
5	0.1	1.6000E-4	2.7122E-4	0.05	1.4000E-4	2.5880E-4
10	0.1	2.7111E-4	1.2287E-3	0.05	3.3000E-4	1.5687E-3
15	0.1	5.2433E-3	9.1122E-3	0.05	6.8001E-3	9.3598E-3
20	0.1	1.0000E-2	2.7741E-2	0.05	1.2600E-2	2.8811E-2

The Interaction of two Solitary Waves:

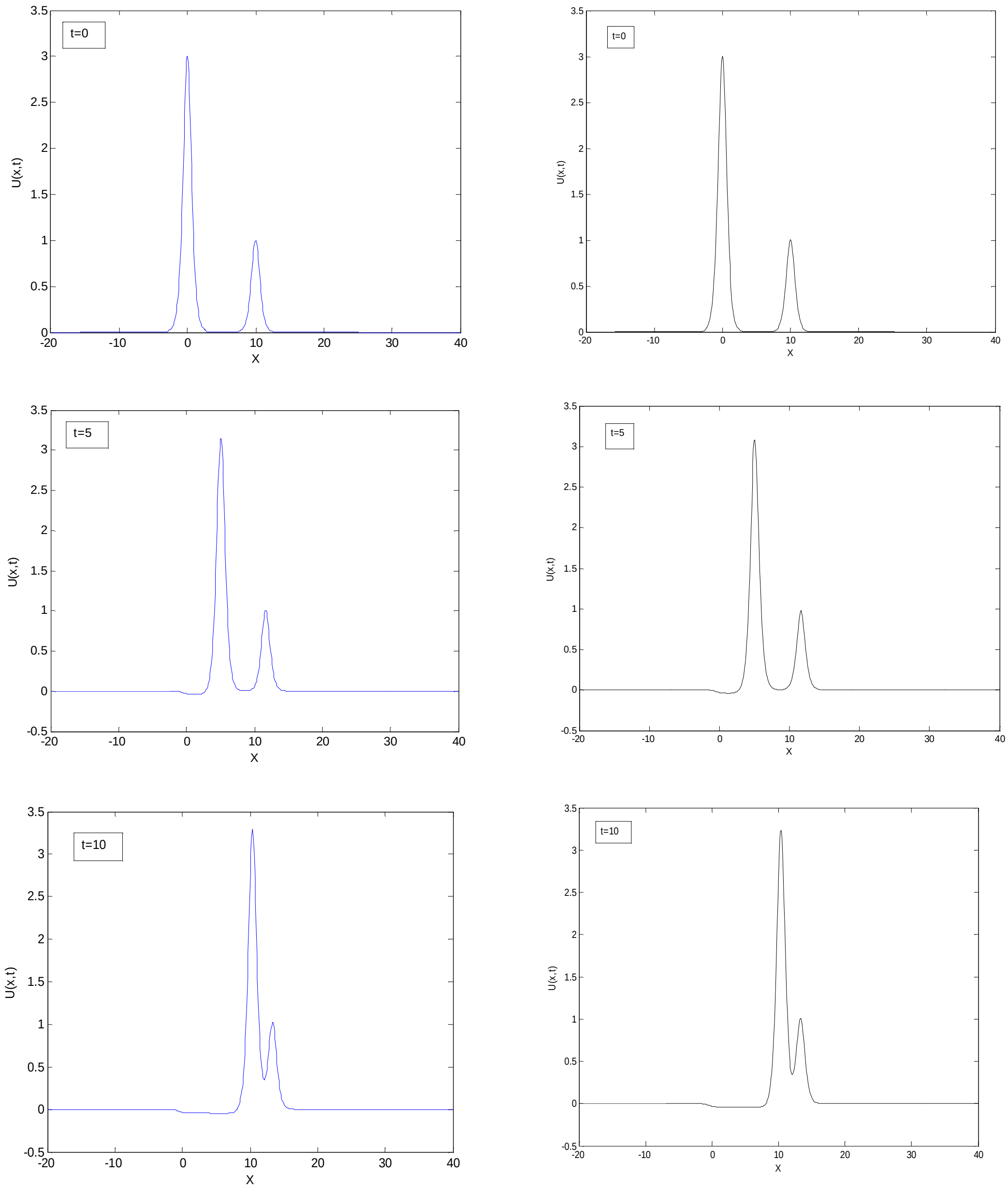
The interaction of two solitary waves for the Coupled BBM-system with the initial condition given by the equations (Chen, 1998) is reported in this section

$$v(x,t) = -1$$

$$u(x,0) = \sum_{i=1}^{2}\left[(1-\frac{g_i}{6})c_i + \frac{c_i g_i}{2}\sec h^2(\frac{\sqrt{g_i}}{x}(x+x_i))\right] \tag{13}$$

Where g_i, c_i and x_i, $i = 1,2$ are real constants. Our system is solved over $-20 \le x \le 40$ with $x_1 = 0, x_2 = -10, c_1 = 1,\quad c_2 = \frac{1}{3}, g_1 = 8, g_2 = 6$, $\Delta t = 0.01$ and $\Delta x = 0.1$. The simulations are executed up to time t=25. In figure 4, the interaction of two solitary waves is shown, the larger amplitude is 3 at x=0 is on the left of the smaller amplitude is 1 at x=10. After the interaction is finished with complete separation at t=25 the amplitude of the larger wave is 3.88 at x=28.5 whereas the amplitude of the second peak is 1.18 at x=17.6 when we used Dirichlet boundary condition In the case of Neumann boundary condition., the amplitude of the larger wave is 3.846 at x=28.7 whereas the amplitude of the second peak is 1.15 at x=17.7, again there is no significant difference between them.

Neumann B. C   Dirichlet B.C



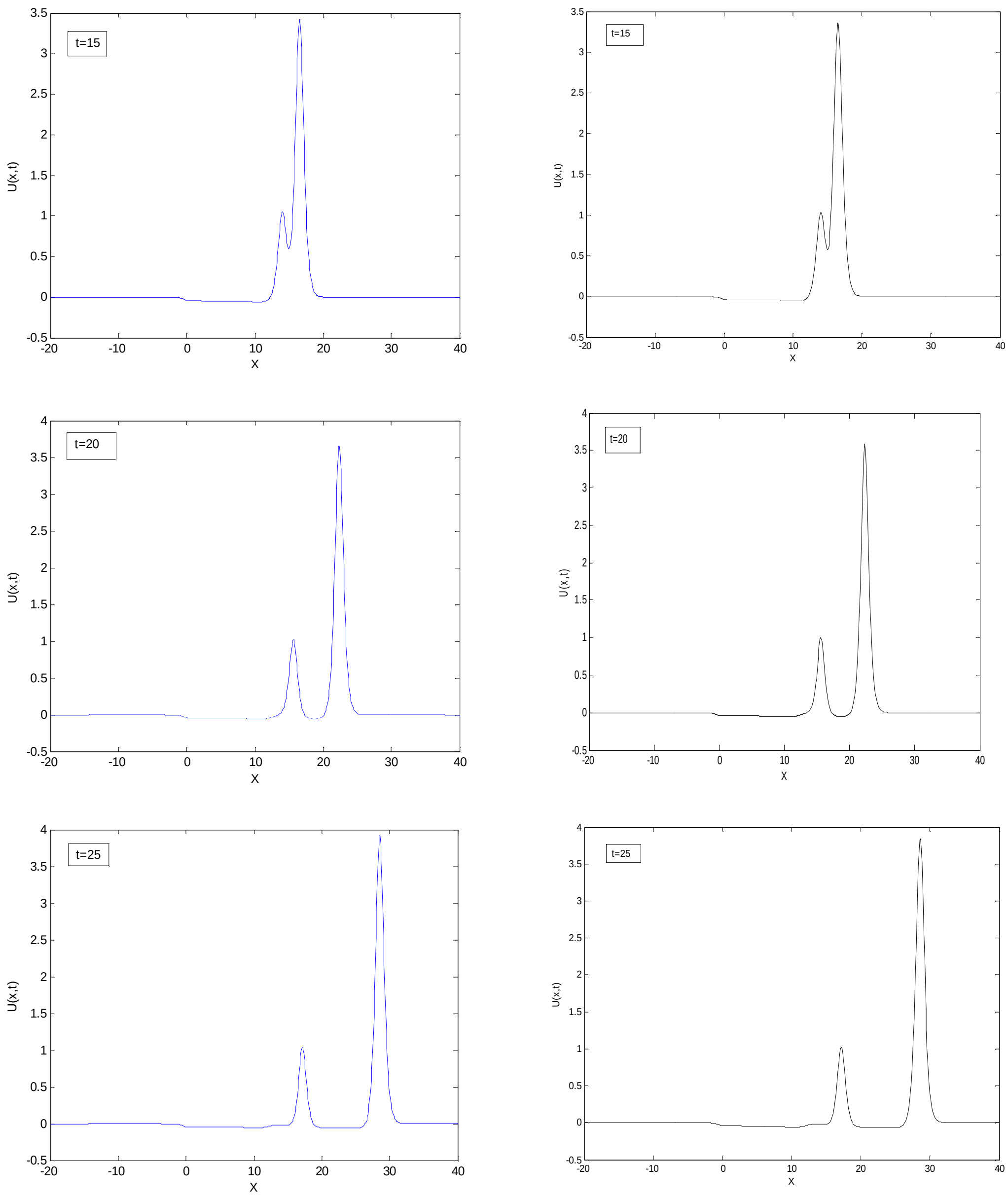


Fig. 4. (a) Fig. 4. (b)

Fig. 4: The interaction of two positive solitary waves at different times ((a) Neumann boundary condition, (b) Dirichlet boundary condition)

Conclusions:

In this paper, a numerical scheme for the nonlinear Coupled BBM-system is presented using Quadratic finite element method. The method has been tested on the motion of single solitary wave and the evolution of two solitary wave interaction. The accuracy of the method was measured using the L_2 error. Numerical results demonstrate the efficiency of the proposed method. From Table 1, it is clear that the results from our method is in good agreement with the exact solution for different values of time and step sizes.

REFERNCES

Bona, J.L. and M. Chen, 1998. A Boussinesq system for two – way propagation of Nonlinear dispersive waves. physica D, 116: 191-224.

Bona, J.L., M. Chen and J.C. Saut, 2004. Boussinesq equations and other Systems for small - amplitude long waves in nonlinear dispersive media. II: The NonlinearTheory, Nonlinearity, 17(3): 925-952.

Bona, J.L., T. Colin and D. Lannes, 2005. Long wave approximations for Water Waves. Archive for rational mechanics and analysis, 178: 373-410.

Chatzipantelidis, P., 1998. Explicit multistep methods for non-stiff partial Differential Equations. Applied Numerical Mathematics, 27: 13-31.

Chen, M., 1998. Exact traveling - wave solutions to bidirectional wave equations. International Journal of theoretical physics, 37(5): 1547-1507.

Chen, M., 2000. Solitary – Wave and multi pulsed traveling wave solutions of Boussinesq systems. Applicable Analysis, 75: 213-240.

Chen, H., M. Chen and N.V. Nguyen, 2007. Cnoidal wave solutions of Boussinesq Systems. Nonlinearity, 20: 1443-1461.

Chen, H. and X. Wang, 2015. Stability of the solitary wave solutions to a coupled BBM system. App. Math.

Dougalis, V.A., D.E. Mitsotakis and J.C. Saut, 2009. On initial boundary Value Problem for Boussinesq system of BBM - BBM type in plane domain. Discrete and continuous dynamical systems, 23: 1191-1204.

Mahmoud, M.A., 2013. Numerical Procedures for Boussinesq type of the Coupled Benjamin-Bona-Mahony system. Ph. D (Thesis), department of Mathematics, University of Mosul. Mosul. Iraq.