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μ-Approximation of Functions by Discrete Operatorsin $L_{p,\mu}(X)$space

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ABSTRACT

In this work, we want to study the error ofμ- approximation of an unbounded function by discrete operators in $L_{p,\mu}(X)$ space, (1≤p< ∞). We intend to establish new theorems regarding by valee-poussin operator and unique linear trigonometric polynomial of these function.

INTRODUCTION

Let $X = [0,1]$,We denote by $L_\infty(X)$ the set of all bounded measurable functions with usual norm,Lorentz (1966)

$$\|f\|_{L_\infty} = \|f\|_\infty = \sup\{|f(x)|,\ x \in X\} \le \infty \tag{1.1}$$

Let $L_p = \{f: f$ is bounded measurable function on $X\}$ for which,Al-Asady (2007)

$$\|f\|_{L_p} = \|f\|_p = \left\{\left(\int_X |f(x)|^p\, dx\right)^{\frac{1}{p}} < \infty\right\},\ \text{where}\ \ 1\le p< \infty \tag{1.2}$$

Further for δ > 0 and (1 ≤ p < ∞) the locally global norm of f is defined by

$$\|f\|_{\delta,p} = (\int_0^1 (\sup\{|f(y)|^p :\ y \in \left[x-\tfrac{\delta}{2}, x+\tfrac{\delta}{2}\right]\} dy)^{\frac{1}{p}} \tag{1.3}$$

Now, let μ be the set of all measurable function on X . Consider $L_{p,\mu}$

$$L_{p,\mu} = \{f: I \longrightarrow R\} \text{and} \|f\|_{p,\mu} = \left\{\left(\int_X |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} < \infty\right\} \tag{1.4}$$

for δ > 0 and (1 ≤ p < ∞) the μ-locally global norm of f is defined by

$$\|f\|_{\delta,p,\mu} = (\int_0^1 (\sup\{|f(y)|^p d\mu(y): y \in \left[x-\tfrac{\delta}{2}, x+\tfrac{\delta}{2}\right]\})^{\frac{1}{p}} \tag{1.5}$$

The kth average modulus of smoothness for f $\in L_p$ with respect to algebraic polynomial for $f \in L_p$ space is defined by, Zygmund (1988)

$$\tau_k(f, \delta)_p = \|\omega_k\ (f, ., \delta)\|_p;\ \ \delta > 0 \tag{1.6}$$

Where

$$\omega_k(f,x,\delta)_p = sup\left\{\left|\Delta_h^k f(t)\right|,\ \ t, t+kh \in \left[x-\tfrac{\delta}{2}, x-\tfrac{\delta}{2}\right], h \le \delta\right\} \tag{1.7}$$

The kth modulus of smoothness for $f \in L_{p,\mu}(X)$ is given by

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$$\omega_k(f,\delta)_{p,\mu} = \underbrace{sup}_{|h|<\delta} \left\{ \left\| \Delta_h^k f(.) \right\|_{p,\mu} \right\} \ , \delta > 0 \tag{1.8}$$

where $\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f\left(x + \frac{kh}{2}\right) \ , x, x+ih \in X$

The k[th] average modulus of smoothness for $f \in L_{p,\mu}(X)$ is defined by:

$$\tau_k(f,\delta)_{p,\mu} = \|\omega_k(f,.,\delta)\|_{p,\mu} = \left(\int_X |\omega_k(f,x,\delta)|^p d\mu(x)\right)^{\frac{1}{p}} \tag{1.9}$$

The K-functional for $f \in X_0$ and $g \in X_1$ is given by

$$K(f,\delta) = K(f,\delta,X_0,X_1) = \underbrace{inf}_{g\in X_1} \{\|f-g\|_{X_0} + \delta\|g\|_{X_1} \ , \delta > 0\} \tag{1.10}$$

Where X_0 and X_1 are two Banach spaces with $X_1 \subset X_0$, Kasim (2004)

The inequality $K(f,\delta) < \epsilon$ for some$\delta > 0$, ϵ is a positive real number, implies that f has approximated with error $\|f-g\| < \epsilon$ in X_0 by an element$g \in X_1$, whose norm is not too large ($\|g\|_{X_1} < \epsilon\,\delta^{-1}$).

The K-functional in $L_p(X)$space is given by,Kasim (2004)

$$K_r(f,\delta^r)_p = \underbrace{inf}_{g\in W_p^r} \left\{\|f-g\|_p + \delta^r \left\|g^{(r)}\right\|_p \ , \delta > 0\right\} \tag{1.11}$$

Where $X_0 = L_p(X)$ and $X_1 = W_p^r$ and $X_1 \subset X_0$,

Now, we introduce K-functional of a function $f \in L_{p,\mu}(X)$ such that,Abdul Naby (2008)

$$K_r(f,\delta^r)_{p,\mu} = \underbrace{inf}_{g\in W_p^r} \left\{\|f-g\|_{p,\mu} + \delta^r \left\|g^{(r)}\right\|_{p,\mu} \ , \delta > 0\right\} \tag{1.12}$$

The degree of µ- best approximation to a function $f \in L_{p,\mu}(X)$ with respect to trigonometric or algebraic polynomials on X is given by

$$E_n(f)_{p,\mu} = inf\{\|f-p_n\|_{p,\mu}; p_n \in \mathbb{P}_n\} \tag{1.13}$$

Where $\mathbb{P}_n$ denote the set of all trigonometric or algebraic polynomials of degree $\leq n$.

The degree of µ-best one-sided approximation of $f \in L_{p,\mu}(X)$ with respect to the trigonometric or algebraic polynomials on X is given by,Deng and Feng (2012)

$$\tilde{E}_n(f)_{p,w} = inf\{\|p_n - q_n\|_{p,w} \ ; p_n, q_n \in \mathbb{P}_n and \ q_n(x) \leq f(x) \leq p_n(x)\} \tag{1.14}$$

Now, consider the Dirichlet kernel of degree n,Zygmand (1988)

$$D_n(u) = \frac{1}{2} + \sum_{i=1}^{n} \cos(iu) \qquad , u \in R \ , \quad n = 1,2,3, \dots \tag{1.15}$$

Let $K_n(u) = \frac{1}{n+1}\,[D_0(u) + D_1(u) + \ \dots + \ D_n(u)]$ (1.16)

be Fejer kernel of degree $\leq n$.

Let $V_{2n}(t) = \frac{1}{n+1}[D_n(t) + D_{n+1}(t) + \ \dots + \ D_{2n}(t)]$,

let$x_j = \frac{2\pi_j}{3n+1}$, j = 0,1,2,…,3n. Then

$$V_{2n,3n}(f,x) = \frac{2}{3n+1}\sum_{j=0}^{3n} f(x_j) V_{2n}(x - x_j) \tag{1.17}$$

be the Valee-Poussin operator.

The unique linear trigonometric polynomial which is interpolating a given function $f \in L_{p,\mu}(X)$ at the point x_i is denoted by $I_n(t)$ which has the following representation :

$$I_n(f,x) = \frac{2}{2n+1}\sum_{i=0}^{2n} f(x_i) D_n(x - x_i). \tag{1.18}$$

Where $x_i = \frac{2\pi i}{2n+1}$,$i = 0,1, \dots ,2n$

Now, let us consider $L_{p,\mu}(X)$ be the set of all unbounded functions, so we consider that $f \in L_{p,\mu}(X)$, $V_{2n,3n}(f) \in L_{p,\mu}(X)$ and $I_n(f) \in L_{p,\mu}(X)$.

2-Basic concept of theorems:

In this section we introduce the results that we make use of.

Lemma 1Abdul Naby (2008):

Let f be a 2π-periodic bounded μ −measurable function then for $1 \leq p < \infty$, we have

$$\tau_r(f,\delta)_{p,\mu} \leq c(p)\delta^r \|f^r\|_{p,\mu}. \tag{2.1}$$

Lemma 2Al-Asady (2007):

Let$f \in L_{p,\mu}$, $1 \leq p < \infty$ and $a, b \in \mathbb{R}$,we have

$$K_r(f,\delta^r)_{p,\mu} \approx \tau_r(f,\delta)_{p,\mu} \tag{2.2}$$

where $r = 1,2, \dots ,$

Lemma 3Lorentz (1966):

If f is a bounded measurable function on the interval [a,b], a,b∈ℝ , then

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^n f(x_i) \quad where\ x_i = a + \frac{(b-a)(2i-1)}{2n}. \tag{2.3}$$

Lemma 4Deng and Feng (2012):

If f is bounded measurable function, then for ($0 < p < 1$) and $\delta > 0$, we have

$$\|f\|_{\delta,p} \leq c(p)(1+n\delta)^{1-p}(ns)^{\frac{1}{p}}\|f\|_p . \tag{2.4}$$

Lemma 5Al-Asady (2007):

Let f be a 2π-periodic μ −measurable function then for $\frac{1}{2(\ell+1)} \leq p < \infty$, we have

$$\|f - I_n(f,x)\|_{p,\mu} \leq c(p,r,\ell)\tau_r(f,\tfrac{1}{n})_{p,\mu} \tag{2.5}$$

Lemma 6Al-Abdull (2005):

Let f be a 2π-periodic μ −measurable function then for $1 \leq p < \infty$, we have

$$\left\|f - V_{2n,4n}(f)\right\|_{p,\mu} \leq c(p)\omega_r^{\varphi}(f,(2n)^{-1})_{p,\mu} \tag{2.6}$$

Lemma (A):

Let $f \in L_{p,\mu}(X)$, then for ($1 \leq p < \infty$) and $\delta > 0$.Then

$$\|f\|_{p,\mu} \leq \|f\|_{\delta,p,\mu} \leq c(p)(1+n\delta)^p\|f\|_{p,\mu}.$$

Proof : We have

$$\|f\|_{p,\mu} = \left(\int_0^1 |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^1 \sup\{|f(x)|^p\}\, d\mu(x)\right)^{\frac{1}{p}}$$

$$= \left(\int_0^1 (\sup\{|f(y)|^p d\mu(y) : y \in \left[x - \tfrac{\delta}{2}, x + \tfrac{\delta}{2}\right]\})\right)^{\frac{1}{p}} = \|f\|_{\delta,p,\mu}.$$

We need to prove that

$$\|f\|_{\delta,p,\mu} \leq c(p)(1+n\delta)^p\|f\|_{p,\mu}$$

We have

$$\|f\|_{\delta,p,\mu} = \left(\int_0^1 (\sup\{|f(y)|^p d\mu(y) : y \in \left[x - \frac{\delta}{2}, x + \frac{\delta}{2}\right]\})\right)^{\frac{1}{p}}$$

By using **(2.3)** we get

$$= \left(\frac{1}{n}\sum_{i=1}^n \sup\left\{|f(y_i)|^p d\mu(y_i) : y_i \in [x_i - \frac{\delta}{2}, x_i - \frac{\delta}{2}]\right\}\right)^{\frac{1}{p}}$$

$$sup\{f(y)d\mu(y)\} = f(y_i)d\mu(y_i)$$

Thus by using **(2.4)** we get

$$\|f\|_{\delta,p,\mu} \leq \left(\frac{1}{n}\sum_{i=0}^n |f(y_i)|^p\right)^{\frac{1}{p}} = \left(\frac{c(p)}{n}\sum_{i=0}^n |f(y_i)|^p\right)^{\frac{1}{p}}$$

$$= c(p)\left(\int_0^1 |f(y)|^p\, d\mu(y)\right)^{\frac{1}{p}}$$

$$= c(p)\|f\|_{p,\mu} \leq c(p)(1+n\delta)^p\|f\|_{p,\mu}$$, for $\delta > 0$ and $1 \leq p < \infty$

we get $\|f\|_{\delta,p,\mu} \leq c(p)(1+n\delta)^p\|f\|_{p,\mu}$.

Lemma (B):

Let $f \in L_{p,\mu}(X)$, be a 2π −periodic unbounded function then for ($1 \leq p < \infty$) and $\delta > 0$, we have $\|f - I_n(f)\|_{p,\mu} \leq c(p,r,\ell)K_r(f,\delta^r)_{p,\mu}$.

Proof: From **(2.5)**and **(2.1)** and since $\delta = \frac{1}{n}$we have

$$\|f - I_n(f)\|_{p,\mu} = \left(\int_0^1 |(f - I_n(f))(x)|^p\, d\mu(x)\right)^{\frac{1}{p}}$$

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$$\leq \left(\int_0^1 |(f - g)(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_0^1 |(g - I_n(g))(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$
$$+ \left(\int_0^1 |(I_n(g) - I_n(f))(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$
$$= \|f - g\|_{p,\mu} + \|g - I_n(g)\|_{p,\mu} + \|I_n(g) - I_n(f)\|_{p,\mu}$$
$$= \|f - g\|_{p,\mu} + \|g - I_n(g)\|_{p,\mu} + \|I_n(g - f)\|_{p,\mu}$$
$$\leq \|f - g\|_{p,\mu} + c_1(p, r, \ell)\tau_r(g, \tfrac{1}{n})_{p,\mu} + c_2(p)\|g - f\|_{p,\mu}$$
$\leq c_3(p)\|f - g\|_{p,\mu} + c_4(p)\delta^r\|g^r\|_{p,\mu}$, where $\delta = \frac{1}{n}$
$$= c(p)\{\|f - g\|_{p,\mu} + \delta^r\|g^r\|_{p,\mu}\} = c(p)K_r(f, \delta^r)_{p,\mu}$$

Hence
$\|f - I_n(f)\|_{p,\mu} \leq c(p)K_r(f, \delta^r)_{p,\mu}$.
Where g is a best approximation

Lemma (C):

Let $f \in L_{p,\mu}(X)$, be a 2π −periodic unbounded function then for ($1 \leq p < \infty$) and $\delta > 0$, we have
$\left\|f - V_{2n,3n}(f)\right\|_{p,\mu} \leq c(p)K_r(f, \delta^r)_{p,\mu}$.

Proof: From **(2.6)** and **(2.1)** and since $\delta = \frac{1}{n}$ we have

$$\left\|f - V_{2n,3n}(f)\right\|_{p,\mu} = \left(\int_0^1 \left|(f - V_{2n,3n}(f))(x)\right|^p d\mu(x)\right)^{\frac{1}{p}}$$
$$\leq \left(\int_0^1 |(f - g)(x)|^p \, d\mu(x)\right)^{\frac{1}{p}} + \left(\int_0^1 \left|(g - V_{2n,3n}(g))(x)\right|^p d\mu(x)\right)^{\frac{1}{p}}$$
$$+ \left(\int_0^1 \left|(V_{2n,3n}(g) - V_{2n,3n}(f))(x)\right|^p d\mu(x)\right)^{\frac{1}{p}}$$
$$= \|f - g\|_{p,\mu} + \left\|g - V_{2n,3n}(g)\right\|_{p,\mu} + \left\|V_{2n,3n}(g) - V_{2n,3n}(f)\right\|_{p,\mu}$$
$$= \|f - g\|_{p,\mu} + \left\|g - V_{2n,3n}(g)\right\|_{p,\mu} + \left\|V_{2n,3n}(g - f)\right\|_{p,\mu}$$
$$\leq \|f - g\|_{p,\mu} + c_1(p)\omega_r^{\varphi}(g, (2n)^{-1})_{p,\mu} + c_2(p)\|g - f\|_{p,\mu}$$
$$\leq c_3(p)\|f - g\|_{p,\mu} + c_4(p)\omega_r(g, (2n)^{-1})_{p,\mu}$$
$\leq c_5(p)\|f - g\|_{p,\mu} + c_6(p)\tau_r(g, \delta)_{p,\mu}$, where $\delta = (2n)^{-1}$
$$\leq c_7(p)\|f - g\|_{p,\mu} + c_8(p)\delta^r\|g^r\|_{p,\mu}$$
$$= c(p)\{\|f - g\|_{p,\mu} + \delta^r\|g^r\|_{p,\mu}\} = c(p)K_r(f, \delta^r)_{p,\mu}$$

Hence
$\left\|f - V_{2n,3n}(f)\right\|_{p,\mu} \leq c(p)K_r(f, \delta^r)_{p,\mu}$. where g is a best approximation

3- Main Results:

In this section we introduce our main results

Theorem (3.1):

Let $f \in L_{p,\mu}(X)$, for ($1 \leq p < \infty$) . Then

$$\tau_k(f, \tfrac{1}{n})_{p,\mu} \leq \tau_k(f, \tfrac{1}{n})_{\delta,p,\mu} \ .$$

Proof: $\tau_k(f, \delta)_{p,\mu} = \|\omega_k(f, ., \delta)\|_{p,\mu}$
$$= \left\|sup\left\{\left|\Delta_h^k f(t)\right| : t, t + kh \in \left[x - \frac{kh}{2}, x - \frac{kh}{2}\right], h \leq \delta\right\}\right\|_{p,\mu}$$
$$= \left(\int_0^1 \sup \left\{\left|\sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(t + ih)\right|^p d\mu(t), t, t + kh \in [x - \frac{k}{2n}, x - \frac{k}{2n}]\right\}\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^1 \sup \left\{ \sup \left\{ \left| \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} f(t+ih) \right|^p d\mu(t+ih), t, t+kh \in \left[x - \frac{k}{2n}, x - \frac{k}{2n} \right] \right\} \right\} d\mu(t) \right)^{\frac{1}{p}}$$

$$= \left\| sup \left\{ \left| \Delta_h^k f(t) \right| : t, t+kh \in \left[x - \frac{kh}{2}, x - \frac{kh}{2} \right], h \leq \delta \right\} \right\|_{\delta,p,\mu}$$

$$= \|\omega_k(f,.,\delta)\|_{\delta,p,\mu} = \tau_k(f,\delta)_{\delta,p,\mu}$$

Corollary (3.2):

Let $f \in L_{p,\mu}(X)$, for $(1 \leq p < \infty)$. Then

$$K_r(f,\delta^r)_{p,\mu} \leq \tau_k(f,\delta)_{\delta,p,\mu}$$

Proof: From **(2.2)** and **theorem (3.1)**, we get

$$K_r(f,\delta^r)_{p,\mu} \approx \tau_r(f,\delta)_{p,\mu} \leq \tau_k\left(f,\frac{1}{\mathrm{n}}\right)_{\delta,p,\mu}$$

Where $\delta = \frac{1}{\mathrm{n}}$

Theorem (3.3):

Let $f \in L_{p,\mu}(X)$, for $(1 \leq \mathrm{p} < \infty)$ and $\delta > 0$.Then

$\left\| f - \mathrm{V}_{2\mathrm{n},3\mathrm{n}}(f) \right\|_{\delta,\mathrm{p},\mu} \leq \mathrm{c(p)}\tau_\mathrm{k}(f,\frac{1}{\mathrm{n}})_{\delta,\mathrm{p},\mu}$.

Proof: From **lemma (A)**, **Lemma (C)**, and **Corollary (3.2)**,we get

$$\left\| f - V_{2n,3n}(f) \right\|_{\delta,p,\mu} \leq c(p)(1+n\delta)^p \left\| f - V_{2n,3n}(f) \right\|_{p,\mu}$$

Since $\delta > 0$, and $\delta = \frac{1}{n}$then

$$\left\| f - V_{2n,3n}(f) \right\|_{\delta,p\mu} \leq c_1(p) \left\| f - V_{2n,3n}(f) \right\|_{p,\mu}$$
$$\leq c_2(p) K_r(f,\delta^r)_{p,\mu}$$
$$\leq c(p,k)\tau_k\left(f,\frac{1}{n}\right)_{\delta,p,\mu}$$

Hence $\left\| f - \mathrm{V}_{2\mathrm{n},3\mathrm{n}}(f) \right\|_{\delta,\mathrm{p},\mu} \leq \mathrm{c(p)}\tau_\mathrm{k}(f,\frac{1}{\mathrm{n}})_{\delta,\mathrm{p},\mu}$.

Theorem (3.4):

Let $f \in L_{p,\mu}(X)$, for $(1 \leq \mathrm{p} < \infty)$ and $\delta > 0$. Then

$\|f - \mathrm{I_n}(f)\|_{\delta,\mathrm{p},\mu} \leq c(p,r,\ell)\tau_\mathrm{k}(f,\frac{1}{\mathrm{n}})_{\delta,\mathrm{p},\mu}$.

Proof: From **Lemma (A)**,**Lemma (B)**, and **Corollary (3.2)**, we get

$$\|f - I_n(f)\|_{\delta,p,\mu} \leq c(p)(1+n\delta)^p \|f - I_n(f)\|_{p,\mu}$$

Since $\delta > 0$, then

$$\|f - I_n(f)\|_{\delta,p,\mu} \leq c_1(p)\|f - I_n(f)\|_{p,\mu}$$
$$\leq c(p,r,\ell)K_r(f,\delta^r)_{p,\mu} \leq c(p,r,\ell)\tau_k\left(f,\frac{1}{n}\right)_{\delta,p,\mu}$$

Hence $\|f - \mathrm{I_n}(f)\|_{\delta,\mathrm{p},\mu} \leq c(p,r,\ell)\tau_\mathrm{k}(f,\frac{1}{\mathrm{n}})_{\delta,\mathrm{p},\mu}$.

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