Numerical Integration of Arbitrary Functions over a Convex and Non Convex Polygonal Domain by Eight Noded Linear Quadrilateral Finite Element Method

A.M. Yogitha and K.T. Shivaram

1Assistant professor, Department of Mathematics, City Engineering College, Bangalore- 62, Karnataka, India.
2Assistant professor, Department of Mathematics, Dayananda Sagar College of Engineering, Bangalore- 78, Karnataka, India.

Address For Correspondence:
K.T. Shivaram, Assistant professor, Department of Mathematics, Dayananda Sagar College of Engineering, Bangalore 560078, Karnataka, India.

ARTICLE INFO
Article history:
Received 11 September 2016
Accepted 10 November 2016
Published 28 November 2016

Keywords:
Numerical Integration, Generalized Gaussian quadrature rule, Convex and Non convex polygonal domain.

ABSTRACT
Objectives: In this paper, the numerical evaluation of the integral of arbitrary function over a convex and Non convex polygonal domain is been investigated.

Method: Convex and Non convex polygonal domain is discretized into sub polygons by eight noded linear quadrilateral Finite element method based on Generalized Gaussian quadrature rule, to evaluate the typical integral of arbitrary functions over the convex and Non convex polygonal domain. Also plotted the extracted sampling points in both convex and non convex region. Numerical examples are given of testing the performance of the proposed method.

INTRODUCTION

As a widely used numerical method, Finite element method is efficient for computer aided engineering, finite element analysis (FEA) has become an integral and major component in the design or modeling of a physical phenomenon in various disciplines. The finite element method has proved superior to other numerical methods due to its better adaptability to any complex geometry. The finite element method (FEM) is a numerical procedure that can be used to obtain solutions to a large class of engineering problems in stress analysis, heat transfer, electromagnetism and fluid flow etc. The calculation of integrals of arbitrary functions over convex and Non convex polygonal domain is one of the most difficult part in solving applied problems in CFD, electrodynamics, quantum mechanics, heat flow across a boundary between materials with different conductivity etc. especially to solving two dimensional partial differential in convex and Non convex region by explicit integration method to extract the stiffness matrix. The integrals arising in practical problems are not always simple and other quadrature scheme cannot evaluate exactly. Numerical Integration over triangle and square region are simple but convex and non convex polygonal domain is challenging task to integration over a arbitrary function. The integration points have to be increased in order to improve the integration accuracy and is desirable to make these evaluations by few gauss points as possible. The method proposed here is termed as Generalized Gaussian quadrature rule.

From the literature of review we may realize that several works in numerical integration using Gaussian quadrature over various regions have been carried out (O.C. Zienkiewicz, R.L. Taylor and J. Z. Zhu, 2000, K. J. Bathe, 1996, Md. Shafiqual Islam and M. Alamgir Hossain, 2009, R.D. Cook, 1981, C. T. Reddy, D. J.

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Shippy, 1981), Gaussian Legendre quadrature over two-dimensional triangle region given in (D.A. Dunavant, 1985, K.T. Shivaram, 2013, G. Lague, R. Baldur, 1977) to construct the numerical algorithm based on optimization and group theory to compute quadrature rule for numerical integration of bivariate polygonal over polygonal domain are presented in (S.E. Mousavi, H. Xiao and N. Sukumar, 2009). Numerical integration over polygonal region by using, spline finite element method and cubature formula over polygons are discussed in (Chong-Jun Li, Paola Lambert and Catterina Dagnino, 2009). In this paper, convex and Non convex polygonal domain is discretized into sub polygons by 8-node quadrilateral elements and then we apply Generalised Gaussian quadrature method to evaluate the typical integral of arbitrary functions over the convex and Non convex polygonal domain.

The paper is organized as follows. In section 2. we present the Generalized Gaussian Quadrature formula. In section 3. presents the mathematical preliminaries required for understanding the derivation and discretized the convex and Non convex polygonal domain into sub 8-node quadrilateral elements and then derive a new Gaussian quadrature formula over a quadrilateral region to calculate sampling points and weight coefficients of various order and also plotted the extracted sampling points in both convex and non convex region. In section 4. we compare the numerical results with some illustrative examples

2. Generalised Gaussian Quadrature rule:

Suppose the integral of the form

$$\int_a^b f(x)\Omega(x)dx = \sum_{i=1}^n w_i^n \Omega(x_i^n)$$

Where $x_i^n \in [a,b]$ and $w_i^n \in \mathbb{R}, \forall i = 1,2,3,4,---n$

$$\Omega(x) = \ln x, \ln x, x^{1/2}, x, x^2, x^3, \cdots$$

Suppose the integral of the form

$$\int_0^1 f(x)\Omega(x)dx = \sum_{i=1}^n w_i^n \Omega(x_i^n)$$

2. Formulation of integrals over a Linear Convex Quadrilateral Element:

The integral of an arbitrary function $f(x,y)$ over an arbitrary convex quadrilateral region $\Omega$ is given by

$$I = \iint_{\Omega} f(x,y)dx\,dy$$

Let us consider an arbitrary 8-node quadrilateral element in the global coordinate is mapped into 2-square in the local coordinate $(\xi, \eta)$. The isoperimetric coordinate transformation from $(x, y)$ plane to $(\xi, \eta)$ is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{k=1}^{8} N_k(\xi, \eta) \begin{pmatrix} x_k \\ y_k \end{pmatrix}$$

Where $(x_k, y_k), (k=1,2,3,4,5,6,7,8)$ are the vertices of the quadrilateral element in $(x, y)$ plane and $N_k(\xi, \eta)$ denotes the shape function of node $k$ such that

$$N_1(\xi, \eta) = \frac{1}{4} (1-\xi)(1-\eta)(-1-\xi-\eta)$$

$$N_2(\xi, \eta) = \frac{1}{4} (1+\xi)(1-\eta)(-1+\xi-\eta)$$

$$N_3(\xi, \eta) = \frac{1}{4} (1+\xi)(1+\eta)(-1+\xi+\eta)$$

$$N_4(\xi, \eta) = \frac{1}{4} (1-\xi)(1+\eta)(-1-\xi+\eta)$$

$$N_5(\xi, \eta) = \frac{1}{4} (1-\xi^2)(1-\eta)$$

$$N_6(\xi, \eta) = \frac{1}{4} (1+\xi^2)(1-\eta)$$

$$N_7(\xi, \eta) = \frac{1}{4} (1-\xi^2)(1+\eta)$$

$$N_8(\xi, \eta) = \frac{1}{4} (1+\xi^2)(1+\eta)$$

From the Eq.(3) and Eq.(4), we have
Similarly

\[ \frac{\partial y}{\partial \xi} = \sum_{k=1}^{n} \frac{\partial N_k}{\partial \xi} = y_2 \left( \frac{1}{4} \xi (1 - \eta)(-1 + \xi - \eta) + \frac{1}{4} \eta (1 + \xi)(1 - \eta) \right) + y_2 \left( \frac{1}{4} \eta (1 + \xi)(1 + \eta) + \frac{1}{4} \xi (1 + \eta)(1 - \eta) \right) + y_5 \left( \frac{1}{2} \xi (1 + \eta)(1 + \eta) \right) + y_6 \left( \frac{1}{2} - \frac{1}{2} \xi^2 \right) - y_8 (1 - \xi) \eta \]

and

\[ f = \frac{\partial (xy)}{\partial (\xi \eta)} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \]
quadrilateral elements in non convex polygonal domain with vertices are (0, 0.75), (0.25, 0.5), (0.75, 0), (0.25, 0), (0.75, 0.75), (0.75, 0.85), (0.5, 1), (7/8, 5/8) and (1/2, 5/8)

3.1. Numerical Integration over a convex region:

Integral form of Eq. (2) rewritten as

\[ I = \iint_{\Omega_1} f(x, y) \, dx \, dy + \iint_{\Omega_2} f(x, y) \, dx \, dy \]

\[ = \int_{-1}^{1} \int_{-1}^{1} f(x(\xi, \eta), y(\xi, \eta)) \, J \, d\xi \, d\eta \]

\[ = \int_{-1}^{1} \int_{-1}^{1} f(m1(\xi, \eta), n1(\xi, \eta)) \, J1 \, d\xi \, d\eta + \int_{-1}^{1} \int_{-1}^{1} f(m2(\xi, \eta), n2(\xi, \eta)) \, J2 \, d\xi \, d\eta \]

\[ = \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j [ f(m1(\xi_i, \eta_j), n1(\xi_i, \eta_j)) \, J1 + f(m2(\xi_i, \eta_j), n2(\xi_i, \eta_j)) \, J2 ] \]

Where

\[ m1 = 0.02500 \, \xi \, \eta + 0.32500 + 0.07500 \, \xi + 0.27500 \, \eta \]

\[ n1 = -0.13750 \, \xi \, \eta - 0.26250 \, \xi + 0.23750 \, \eta + 0.362500 \]

\[ J1 = -0.08500 - 0.0168750 \, \eta - 0.00343750 \, \eta^2 + 0.0318750 \, \xi + 0.0034375 \, \xi^2 \]

\[ m2 = -0.01250 \, \xi \, \eta + 0.73750 + 0.13750 \, \xi - 0.112500 \, \eta \]

\[ n2 = 0.637500 - 0.112500 \, \xi \, \eta + 0.037500 \xi + 0.28750 \, \eta \]

\[ J2 = -0.1593750 \, \eta + 0.0014062 \, \eta^2 + 0.03750 - 0.0090625 \, \xi - 0.00140625 \, \xi^2 \]

Where \( \xi_i, \eta_j \) are sampling points and \( w_i, w_j \) are corresponding weights.

We present the following algorithm to calculate sampling points and its weight coefficients as:

Case 1

**step 1.** \( k \rightarrow 1 \)

**step 2.** \( i = 1, m \)

**step 3.** \( j = 1, n \)

\[ W_k = J1 \ast w_i \ast w_j \]

\[ x_k = m1 \]

\[ y_k = n1 \]

\[ k = k + 1 \]

**step 4. compute step 3**

**step 5. compute step 2**

Case 2

**step 1.** \( k \rightarrow 1 \)

**step 2.** \( i = 1, m \)

**step 3.** \( j = 1, n \)

\[ W_k = J2 \ast w_i \ast w_j \]

\[ x_k = m2 \]

\[ y_k = n2 \]

\[ k = k + 1 \]

**step 4. compute step 3**

**step 5. compute step 2**

result = Case 1 + Case 2

To computed the sampling points and corresponding weights based on the above algorithm for order \( N = 5, 10, 15, 20 \) and plotted the distribution of sampling points in convex polygonal domain of various order.
3.2. Numerical Integration over a Non convex region:

Integral form of Eq. (2) rewritten as

\[
I = \int_{-1}^{1} \int_{-1}^{1} f(\xi, \eta) d\xi d\eta
\]

\[
= \frac{1}{J1} \int_{-1}^{1} \int_{-1}^{1} f(m1(\xi, \eta), n1(\xi, \eta)) J1 d\xi d\eta + \frac{1}{J2} \int_{-1}^{1} \int_{-1}^{1} f(m2(\xi, \eta), n2(\xi, \eta)) J2 d\xi d\eta + \frac{1}{J3} \int_{-1}^{1} \int_{-1}^{1} f(m3(\xi, \eta), n3(\xi, \eta)) J3 d\xi d\eta
\]

\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} w_i w_j \left[ f(m1(\xi_i, \eta_j), n1(\xi_i, \eta_j)) J1 + f(m2(\xi_i, \eta_j), n2(\xi_i, \eta_j)) J2 + f(m3(\xi_i, \eta_j), n3(\xi_i, \eta_j)) J3 + f(m4(\xi_i, \eta_j), n4(\xi_i, \eta_j)) J4 + f(m5(\xi_i, \eta_j), n5(\xi_i, \eta_j)) J5 \right]
\]

Where

\[
m1 = 0.06250 \xi + 0.187500 \eta + 0.312500 - 0.06250 \eta \xi
\]

\[
n1 = -0.156250 \xi + 0.093750 \eta - 0.031250 \eta \xi + 0.7187500
\]

\[
J1 = 0.0019531 \eta^2 + 0.007812 - 0.02734375 - 0.0019531 \xi^2 + 0.01171875 \xi
\]

\[
m2 = 0.187500 \xi + 0.1250000 \eta - 0.06250 \xi^2 - 0.062500 \eta - 0.187500 \eta \xi^2 + 0.3750000
\]

\[
n2 = 0.1562500 \xi + 0.2812500 \eta - 0.1250000 \xi^2 + 0.0937500 \eta \xi - 0.1250000 \eta + 0.5300000
\]

\[
J2 = 0.031250 \xi^2 - 0.101562500 \eta + 0.000195312 \xi + 0.00625 \eta \xi^2 + 0.000589375 \eta \xi^2 - 0.02734375 \eta + 0.05078125 \eta \xi^2 + 0.093750 \eta \xi^2 - 0.025390625 \xi - 0.0234375 \eta \xi^2 - 0.023437500
\]

\[
m3 = -0.093750 \eta + 0.1875000 \xi^2 + 0.093750 \eta^2 + 0.56250 - 0.0312500 \eta + 0.06250 \eta \xi^2 + 0.093750 \eta \xi^2
\]

\[
n3 = 0.156250 \xi + 0.1562500 \eta + 0.406250 - 0.093750 \eta \xi
\]

\[
J3 = -0.027343750 \eta \xi + 0.054687500 \xi + 0.017578125 \eta^2 - 0.0341796875 \eta - 0.008780625 \xi^3 - 0.011718750 \eta \xi^2 + 0.002695312 \xi^2 + 0.00589375 \xi^3 + 0.017578125 \eta \xi^2 + 0.014648375 \eta
\]

\[
m4 = 0.1250000 \xi - 0.0625000 \eta - 0.0937500 \xi^2 - 0.0312500 \eta^2 + 0.031250 \eta \xi - 0.031250 \eta \xi^2 - 0.0312500 \xi \eta^2 + 0.0843750
\]

\[
n4 = 0.031250 \xi + 0.06250 \eta + 0.06250 \eta \xi^2 + 0.031250 \xi \eta^2 + 0.531250
\]

\[
J4 = 0.021484375 \xi^2 + 0.00195312 \xi \eta^2 + 0.0078125 \xi^2 + 0.0078125 \xi \eta + 0.000976656 \eta^3 + 0.0087890625 \eta^2 - 0.0009766562 \eta \xi^2 + 0.006835937 \eta^2 + 0.0078125 - 0.0078125 \eta \xi^2 - 0.001953125 \eta^3 + 0.009766562 \eta^4 - 0.014648375 \eta \xi^2
\]

\[
m5 = 0.125000 \xi - 0.031250 \eta - 0.031250 \xi \eta^2 + 0.031250 \xi \eta^2 + 0.656250
\]

\[
n5 = -0.031250 \xi + 0.056250 \eta + 0.125 \xi \eta^2 + 0.025 \eta \xi^2 - 0.0687500 \eta \xi + 0.02500 \eta \xi^2 + 0.06250 \eta \xi^2 + 0.656250
\]
\[
J_5 = -0.015429687 \xi^2 + 0.01953125 \eta \xi - 0.005859375 \eta \xi^2 + 0.006250 \eta \xi^2 + 0.031054687 \xi \eta^2 - 0.00156250 \xi^3 - 0.00156250 \xi^3 \eta
\]

Sampling points and corresponding weights are calculated by the above equations for order N = 5, 10, 15, 20 and plotted the distribution of sampling points in Non convex polygonal domain of various order.

Fig. 3: Distribution of Sampling points in Non convex polygonal domain

4. Numerical Results:
We have compared the numerical results obtained using Generalised Gaussian quadrature method with that of numerical results arrived in (Chong-Jun Li, Paola Lamberti and Catterina Dagnino, 2009) of various order N = 5, 10, 15, 20 and are tabulated in Table 1 and 2.

Table 1: Convex region

<table>
<thead>
<tr>
<th>Exact value</th>
<th>Order</th>
<th>Computed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int_C e^{-100(x-0.5)^2+(y-0.5)^2} dxdy )</td>
<td>N=5</td>
<td>0.03141408</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.03141451</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.03141452</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.03141452</td>
</tr>
<tr>
<td>( \int_C \sqrt{(x-0.5)^2+(y-0.5)^2} dxdy )</td>
<td>N=5</td>
<td>0.031568250977</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.15682559</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.15682511</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.15682512</td>
</tr>
<tr>
<td>( \int_C \frac{1}{x^2+y^2-0.25} dxdy )</td>
<td>N=5</td>
<td>0.19906273</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.19906204</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.19906251</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.19906254</td>
</tr>
<tr>
<td>( \int_C \sqrt[3]{3-4x-3y} dxdy )</td>
<td>N=5</td>
<td>0.54538689</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.54538682</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.54538680</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.54538680</td>
</tr>
</tbody>
</table>

Table 2: Non convex region

<table>
<thead>
<tr>
<th>Exact value</th>
<th>Order</th>
<th>Computed value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int_C e^{-100(x-0.5)^2+(y-0.5)^2} dxdy )</td>
<td>N=5</td>
<td>0.03122085</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.03122081</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.03122081</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.03122083</td>
</tr>
<tr>
<td>( \int_C \sqrt{(x-0.5)^2+(y-0.5)^2} dxdy )</td>
<td>N=5</td>
<td>0.13938144</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.13938145</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.13938145</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.13938145</td>
</tr>
<tr>
<td>( \int_C \frac{1}{x^2+y^2-0.25} dxdy )</td>
<td>N=5</td>
<td>0.20842556</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.20842553</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.20842556</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.20842559</td>
</tr>
<tr>
<td>( \int_C \sqrt[3]{3-4x-3y} dxdy )</td>
<td>N=5</td>
<td>0.45453049</td>
</tr>
<tr>
<td></td>
<td>N=10</td>
<td>0.45453053</td>
</tr>
<tr>
<td></td>
<td>N=15</td>
<td>0.45453055</td>
</tr>
<tr>
<td></td>
<td>N=20</td>
<td>0.45453055</td>
</tr>
</tbody>
</table>
Conclusion:

In this paper, a new quadrature method based on Generalised Gaussian quadrature rule is applied for the numerical integration of arbitrary function in convex and non convex polygonal region is discretised into 8 – noded quadrilateral element. We have compared the numerical results obtained using Generalised Gaussian quadrature method with that of numerical results arrived in (Chong-Jun Li, Paola Lamberti and Catterina Dagnino, 2009) of various order $N = 5, 10, 15, 20$. The results obtained are in excellent agreement with exact values.

REFERENCES


