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A Generalization of Weierstrass Inequality with Some Parameters

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Abstract

This study aims to establish new forms of the Weierstras's inequality with some parameters. The Weierstras's inequality is one of the most famous inequality and most useful tool in Mathematical analysis. We use principles of mathematical analysis as the principle of Mean Value Theorem, order relationship properties and other principles to find and generalized new forms of Weierstras's inequality. We hope the results in this paper help other Interested researchers in this area of research.

Keywords: Weierstrass Inequality, Bernoulli's inequality, Mean Value Theorem (MVT)

INTRODUCTION

Suppose $x > -1$ and $n \in \mathrm{N}$. Then

$$(1+x)^n \geq 1+nx \tag{1}$$

(1) Is called Bernoulli's inequality, which is essential in the analysis [4].

Suppose $0 < x_i < 1, i = 1,2,3,\ldots,n$, where $n > 2$. The following inequalities

$$\prod_{i=1}^{n}(1+x_i) \geq 1+\sum_{i=1}^{n} x_i \tag{2}$$

$$\prod_{i=1}^{n}(1-x_i) \geq 1-\sum_{i=1}^{n} x_i \tag{3}$$

are well known as Weierstrass's inequality [1] or Weierstrass's Bernoulli's inequality [3]. These inequalities are two of the most important inequalities in the supject of product polynomials. Many scientists studied this topic and a large number of documents have been written on it [6],[9],[8]. For example, if $\alpha_1 \geq 0, x_i \geq -1, i = 1,2,\ldots,n$, and $\sum_{i=1}^{n} \alpha_i \leq 1$; then

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$$\prod_{i=1}^{n}(1+x_i)^{\alpha_i} \le 1+\sum_{i=1}^{n}\alpha_i x_i \quad (4)$$

If $\alpha_i \ge 1,\ x_i > 0$ or $\alpha_i \le 1$ and $x_i < 0,$ for $i = 1,2,\ldots,n$ then

$$\prod_{i=1}^{n}(1+x_i)^{\alpha_i} \ge 1+\sum_{i=1}^{n}\alpha_i x_i \quad (5)$$

Let $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = 1$; H.Sh. Huang gave an inequality [2]

$$\prod_{i=1}^{n}\left(\frac{1}{x_i}+x_i\right) \ge \left(n+\frac{1}{n}\right)^n \quad (6)$$

More general, for $x_1, x_2, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = k, \quad 1 \le k \le n$ where $k, n \in \mathrm{N}$; for $m \in \mathrm{N}$, Shanhe Wu and Huannan Shi [10] gave the following inequality

$$\prod_{i=1}^{n}\left(\frac{1}{x_i^{\,m}}+x_i^{\,m}\right) \ge \left(\frac{n^m}{k^m}+\frac{k^m}{n^m}\right)^n \left(\prod_{i=1}^{n}\frac{n}{k}x_i\right)^{\frac{m(k^{2m}-n^{2m})}{k^{2m}+n^{2m}}} \ge \left(\frac{n^m}{k^m}+\frac{k^m}{n^m}\right)^n \quad (7)$$

For more information on the weierstrass inequality, you can refer to [5,7] and the references therein .

In this paper, new generalizations of Weierstrass inequality are established by using principles of mathematical analysis as the principle of Mean Value Theorem.

PRELIMINARIES AND LEMMAS

LEMMA1. Let $\alpha_i > 2,\ \beta < 1$ and $0 \le x_i < 1$ for $i = 1,2,\ldots,n$. Then

$$\sum_{i=1}^{n}\beta x_i^{\alpha_i} \ge \prod_{i=1}^{n}\beta x_i^{\alpha_i} \quad (8)$$

PROOF: If $x_i = 0$, then (8) holds.

If $0 < x_i < 1$ and $\alpha_i > 2$ then $(x_i)^{\alpha_i} < 1$ and since $\beta < 1$ then

$$\beta(x_1^{\alpha_1}+x_2^{\alpha_2}+\cdots+x_n^{\alpha_n}) > \beta(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n})$$

thus we have

$$\sum_{i=1}^{n}\beta x_i^{\alpha_i} \ge \prod_{i=1}^{n}\beta x_i^{\alpha_i}.$$

LEMMA2. Let $1 \le \beta \le \gamma_i\ \ \alpha_i > 2,\ \ \gamma^{\frac{1}{\alpha_i - 1}} < \beta$ and $x_i > 1$ for $i = 1,2,\ldots,n$. Then

$$\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_1} > \sum_{i=1}^{n}\gamma_i \beta x_i^{\alpha_i} \quad (9)$$

PROOF: Define

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$$h(x_i) = (\gamma_i + \beta x_i)^{\alpha_i} - \gamma_i \beta x_i^{\alpha_i} \quad \text{for } x_i \geq 1$$

Its clear that $h(x_i)$ is continuous function for $x_i \geq 1$ and differentiable for $x_i > 1$ with

$$h'(x_i) = \alpha_i \beta (\gamma_i + \beta x_i)^{\alpha_i - 1} - \alpha_i \gamma_i \beta x_i^{\alpha_i - 1}$$
$$= \alpha_i \beta (\gamma_i + \beta x_i)^{\alpha_i - 1} \left(1 - \frac{\gamma_i x_i^{\alpha_i - 1}}{(\gamma_i + \beta x_i)^{\alpha_i - 1}} \right)$$

since $\gamma^{\frac{1}{\alpha_i - 1}} x_i < \beta x_i < \gamma_i + \beta x_i$ then $\frac{\gamma^{\frac{1}{\alpha_i - 1}} x_i}{\gamma_i + \beta x_i} < 1$, it is mean that $\left(\frac{\gamma^{\frac{1}{\alpha_i - 1}} x_i}{\gamma_i + \beta x_i} \right)^{\alpha_i - 1} = \frac{\gamma_i x_i^{\alpha_i - 1}}{(\gamma_i + \beta x_i)^{\alpha_i - 1}} < 1$, we obtain that

$h'(x) > 0$. By using MVT, $\exists c_i \in (1, x)$ such that $h'(c_i) = \frac{h(x_i) - h(1)}{x_i - 1}$, then

$$h'(x_i) = \alpha_i \beta (\gamma_i + \beta x_i)^{\alpha_i - 1} \left(1 - \left(\frac{\gamma_i^{\frac{1}{\alpha_i - 1}} c_i}{\gamma_i + \beta c_i} \right)^{\alpha_i - 1} \right)$$
$$= \frac{\left[(\gamma_i + \beta x_i)^{\alpha_i} - \gamma_i \beta x_i^{\alpha_i} \right] - \left[(\gamma_i + \beta)^{\alpha_i} - \gamma_i \beta \right]}{x_i - 1}$$

since $\frac{\gamma_i^{\frac{1}{\alpha_i - 1}} c_i}{\gamma_i + \beta c_i} < 1$, thus

$$h'(x_i) = \alpha_i \beta (\gamma_i + \beta x_i)^{\alpha_i - 1} \left(1 - \left(\frac{\gamma_i^{\frac{1}{\alpha_i - 1}} c_i}{\gamma_i + \beta c_i} \right)^{\alpha_i - 1} \right) > 0,$$

its mean that $h(x_i)$ is increasing function for $x_i \geq 1$, and also by MVT

$$h'(x_i) = \frac{\left[(\gamma_i + \beta x_i)^{\alpha_i} - \gamma_i \beta x_i^{\alpha_i} \right] - \left[(\gamma_i + \beta)^{\alpha_i} - \gamma_i \beta \right]}{x_i - 1} > 0$$

since $x_i > 1$, then

$$\left[(\gamma_i + \beta x_i)^{\alpha_i} - \gamma_i \beta x_i \right] - \left[(\gamma_i + \beta)^{\alpha_i} - \gamma_i \beta \right] > 0 \qquad (10)$$

since $1 \leq \beta \leq \gamma_i < \alpha_i$, then $(\gamma_i + \beta)^{\alpha_i} > \gamma_i \beta$, from (10) we obtain

$$(\gamma_i + \beta x_i)^{\alpha_i} > \gamma_i \beta x_i^{\alpha_i} \qquad (11)$$

then

$$\prod_{i=1}^{n} (\gamma_i + \beta x_i)^{\alpha_i} > \prod_{i=1}^{n} \gamma_i \beta x_i^{\alpha_i} \qquad (12)$$

and since $\gamma_i \beta x_i > 2$ for $i = 1, 2, \ldots, n$, then

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$$\prod_{i=1}^{n}\gamma_i\beta x_i^{\alpha_i} > \sum_{i=1}^{n}\gamma_i\beta x_i^{\alpha_i} \tag{13}$$

Apply transitive property to (12) and (13) we obtain (9).

MAIN RESULTS

THEOREM 1. Let $1\le\beta\le\gamma_i$, $\alpha_i>2$ and $0\le x_i<1$, for $i=1,2,\ldots,n$, then

$$\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}\ge\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}+\sum_{i=1}^{n}\beta x_i^{\alpha_i} \tag{14}$$

and

$$\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}\ge\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}+\prod_{i=1}^{n}\beta x_i^{\alpha_i} \tag{15}$$

PROOF: If $x_i=0$ (14) holds.

For $0<x_i<1$, define

$$f(x_i)=\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}-\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}-\sum_{i=1}^{n}\beta x_i^{\alpha_i} \tag{16}$$

then $f(x_i)$ is differentiable function with

$$\begin{aligned} f'(x_i)&=\sum_{i=1}^{n}\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\ne i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\sum_{i=1}^{n}\alpha_i\beta x_i^{\alpha_i-1}\\ &=\sum_{i=1}^{n}\left(\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\ne i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-(\alpha_i\beta x_i^{\alpha_i-1})\right)\end{aligned} \tag{17}$$

since $\gamma_i+\beta x_i>x_i$ and since $\alpha_i-1>0$, then $(\gamma_i+\beta x_i)^{\alpha_i-1}>x_i^{\alpha_i-1}$, and since $(\gamma_i+\beta x_i)^{\alpha_i}>1$ then $\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}>1$, thus we obtain

$$\left(\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\ne i}}^{n}(\gamma_i+\beta x_i)^{\alpha_i}\right)-\alpha_i\beta x_i^{\alpha_i-1}\right)>0 \tag{18}$$

then

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$$\sum_{i=1}^{n}\left(\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\alpha_i\beta x_i^{\alpha_i-1}\right)>0 \qquad (19)$$

thus $f'(x_i)>0$ for $0<x_i<1$, then $f(x_i)$ is increasing for $0\le x_i<1$, its mean that $f(x_i)>f(0)=0$, thus

$$f(x_i)=\left(\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}-\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}-\sum_{i=1}^{n}\beta x_i^{\alpha_i}\right)>0 \qquad \text{for } x\in[0,1)$$

then

$$\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}>\left(\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}+\sum_{i=1}^{n}\beta x_i^{\alpha_i}\right)>0$$

and because the inequality (14) holds when $x_i=0$, then

$$\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}\ge\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}+\sum_{i=1}^{n}\beta x_i^{\alpha_i}>0$$

Which complete the proof of (14) of theorem 1.

To prove (15) : If $x_i=0$ (15) holds

For $0<x_i<1$, define

$$g(x_i)=\prod_{i=1}^{n}(\gamma_i+\beta x_i)^{\alpha_i}-\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}-\prod_{i=1}^{n}\beta x_i^{\alpha_i} \qquad (20)$$

Its clear that $g(x_i)$ is continuous for $0\le x_i<1$ and differentiable function with

$$g'(x_i)=\sum_{i=1}^{n}\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\sum_{i=1}^{n}\left(\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_j^{\alpha_j}\right) \qquad (21)$$

$$g'(x_i)=\sum_{i=1}^{n}\left(\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_i^{\alpha_j}\right) \qquad (22)$$

since $(\gamma_i+\beta x_i)^{\alpha_i-1}>x_i^{\alpha_i-1}$ and $(\gamma_i+\beta x_i)^{\alpha_i}>\beta x_i^{\alpha_i}$, then

$$\left[\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\left(\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_j^{\alpha_j}\right)\right]>0 \qquad (23)$$

then

$$\sum_{i=1}^{n}\left[\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\left(\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_j^{\alpha_j}\right)\right]>0 \qquad (24)$$

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then $g'(x_i) > 0$ for $0 < x_i < 1$, then $g(x_i)$ is increasing on $0 \le x_i < 1$, its mean that $g(x_i) > g(0) = 0$, then we lead the proof of inequality (15).

Remark 1:If we put $\gamma_i = 1,\ \beta = 1$ and $\alpha_i = 1$ in (14), we obtain the famous Weierstrass's inequality (2).

Remark 2:If we put $\gamma_i = 1,\ \beta = 1$ and $\alpha_i = \alpha$ in (14), we obtain the famous Weierstrass's inequality (4).

THEOREM 2: Let $1 \le \gamma_i < \alpha_i,\ \beta > 2$ and $x_i = 1$, then

$$\prod_{i=1}^{n} (\gamma_i + \beta x_i)^{\alpha_i} \ge \prod_{i=1}^{n} (\gamma_i)^{\alpha_i} + \sum_{i=1}^{n} \beta x_i^{\alpha_i}$$

(25)

Proof: Substitute $x_i = 1$ in the left hand side of (25) and let

$$I = \prod_{i=1}^{n} (\gamma_i + \beta)^{\alpha_i} = (\gamma_1 + \beta)^{\alpha_1} (\gamma_2 + \beta)^{\alpha_2} (\gamma_3 + \beta)^{\alpha_3} \ldots (\gamma_n + \beta)^{\alpha_n}$$

Since $1 \le \gamma_i < \alpha_i$, for $i = 1,2,\ldots,n$, then

$$I \ge (\gamma_1^{\alpha_1} + \beta^{\alpha_1})(\gamma_2^{\alpha_2} + \beta^{\alpha_2})(\gamma_3^{\alpha_3} + \beta^{\alpha_3})\ldots(\gamma_n^{\alpha_n} + \beta^{\alpha_n})$$
$$= (\gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \gamma_3^{\alpha_3} \ldots \gamma_n^{\alpha_n}) + \underbrace{C_1(\gamma_i^{\alpha_i}, \beta^{\alpha_i})}_{..>1} + \underbrace{C_2(\gamma_i^{\alpha_i}, \beta^{\alpha_i})}_{>1} + \ldots + \underbrace{C_{2^n=2}(\gamma_i^{\alpha_i}, \beta^{\alpha_i})}_{>1} + \beta^{\alpha_1} \beta^{\alpha_2} \beta^{\alpha_3} \ldots \beta^{\alpha_n}$$

(26) Where $C_j,\ j = 1,2,\ldots,2^{n-2}$ are coefficients of γ_i and β, and since all of $C_j > 1$ for $j = 1,2,\ldots,2^{n-2}$, then

$$I \ge \prod_{i=1}^{n} \gamma_i^{\alpha_{i1}} + \sum_{j=1}^{2^n-2} \underbrace{C_j(\gamma_i^{\alpha_i}, \beta^{\alpha_i})}_{..>1} + \prod_{i=1}^{n} \beta^{\alpha_i}$$

$$\ge \prod_{i=1}^{n} \gamma_i^{\alpha_{i1}} + \prod_{i=1}^{n} \beta$$

Which complete the proof of theorem 2.

THEOREM 3. Let $1 \le \beta \le \gamma_i < \alpha,\ \gamma_i^{\frac{1}{\alpha_i - 1}} x_i < (\gamma_i + \beta x_i),\ \gamma_i \beta x_i > 2$ and $x_i > 1$ for $i = 1,2,\ldots,n$, then

$$\prod_{i=1}^{n} (\gamma_i + \beta x_i)^{\alpha_i} > \prod_{i=1}^{n} (\gamma_i)^{\alpha_i} + \prod_{i=1}^{n} \beta x_i^{\alpha_i} \tag{27}$$

Proof: Define

$$g(x_i) = \prod_{i=1}^{n} (\gamma_i + \beta x_i)^{\alpha_i} - \prod_{i=1}^{n} (\gamma_i)^{\alpha_i} - \prod_{i=1}^{n} \beta x_i^{\alpha_i}$$

then $g(x_i)$ is differentiable function with

$$g'(x_i) = \sum_{i=1}^{n} \left(\alpha_i \beta (\gamma_i + \beta x_i)^{\alpha_i - 1} \prod_{\substack{j=1 \\ j \ne i}}^{n} (\gamma_j + \beta x_j)^{\alpha_j} \right) - \sum_{i=1}^{n} \left(\alpha_i \beta x_i^{\alpha_i - 1} \prod_{\substack{j=1 \\ j \ne i}}^{n} \beta x_j^{\alpha_j} \right)$$

$$=\sum_{i=1}^{n}\left(\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_i^{\alpha_j}\right)$$

since $(\gamma_i+\beta x_i)^{\alpha_i-1}>\beta x_i^{\alpha_i-1}$ and using Lemma 2, we get

$$\left[\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\left(\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_j^{\alpha_j}\right)\right]>0$$

then

$$\sum_{i=1}^{n}\left[\left(\alpha_i\beta(\gamma_i+\beta x_i)^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}(\gamma_j+\beta x_j)^{\alpha_j}\right)-\left(\alpha_i\beta x_i^{\alpha_i-1}\prod_{\substack{j=1\\ j\neq i}}^{n}\beta x_j^{\alpha_j}\right)\right]>0$$

then $g'(x_i)>0$, for $x_i>1$, then $g(x_i)$ is incrasing for $x_i\geq 1$, thus

$$g(x_i)>g(1)=\left(\prod_{i=1}^{n}(\gamma_i+\beta)^{\alpha_i}-\prod_{i=1}^{n}(\gamma_i)^{\alpha_i}-\sum_{i=1}^{n}\beta\right)>0$$

for $x_i\in(1,\infty)$, which leads to inequality(27).

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