

## Note On Random Recursive Trees

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**Abstract:** Meir and Moon (1988) obtained a convergence in probability for outdegree in random recursive trees. In this paper the result of Meir and Moon is extended to a.s convergence.

**Keywords:** Outdegree, a.s convergence, tree, urn

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### INTRODUCTION

Some results for random recursive trees is obtained. For example, Mahmoud and Smythe (1992) obtained a asymptotic joint normality of outdegrees of node in random recursive trees. Meir and Moon (1988) obtained a convergence in probability for outdegree in random recursive trees. They showed that if  $Y_{ni}$  be the number of vertices of outdegree  $i=0,1,2$ , in a random recursive trees with  $n$  vertices, then as  $n \rightarrow \infty$ ,

$$\frac{Y_{ni}}{n} \xrightarrow{P} \left(\frac{1}{2}\right)^{i+1} \quad (1)$$

Na and Rapoport (1970) studied the expected number of nodes of degree two. They showed that

$$E\left(\frac{Y_{ni}}{n}\right) \rightarrow \left(\frac{1}{2}\right)^{i+1}. \quad (2)$$

Najock and Heyde (1982) found the exact and asymptotic distribution of leaves. Javanian and Vahidi-Asl (2003) found the exact probability distribution of the outdegree of the node  $i$  in a random recursive trees with  $n$  nodes.

A tree is a connected graph with no cycles. A rooted recursive tree of order  $n$  is a tree on  $n$  vertices (denoted by  $T_n$ ) labeled  $1,2,3,\dots,n$ , with the node labeled 1 distinguished as the root, and such that for each  $k$ ,  $2 \leq k \leq n$ , the labels of the vertices in the unique path jointing the root to vertex labeled  $k$  from an increasing sequence. By convention, within any level the immediate children of a node must be numbered from left to right in an increasing sequence.

#### Definition:

A random recursive tree of order  $n$  is one chosen with equal probability from the space of all such that trees. Hence, this model has been termed the uniform recursive tree.

It is an easy induction to show that there are  $(n-1)!$  recursive trees of order  $n$  (Mahmoud and Smythe 1992). Thus each random tree of order  $n$  occurs with probability  $1/(n-1)!$ . There exists a simple growth rule for uniform recursive trees under which the addition of node to make the transition from a random tree of size  $n-1$  to a random tree of size  $n$  will appear as if the tree of order  $n$  has been picked at random from its sample space: A random recursive tree  $T_n$  of order  $n$  as evolving from the recursive trees  $T_{n-1}$  of order  $n-1$ , by choosing a node (parent) of  $T_{n-1}$  at random and joining a node labeled  $n$  (child) to it, all  $n-1$  nodes of  $T_{n-1}$  being equally likely.

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Figure 1 shows all random recursive trees of order 4. Recursive trees have been proposed as models for the spread of epidemics; for pyramid schemes; for the family trees of preserved copies of ancient or medieval texts; and for algorithms used to produce convex hulls in higher dimensions [ Mahmoud and Smythe (1992)]. The *outdegree* of node in a random recursive tree is the number of immediate descendants of that node. The nodes of outdegree zero, usually called the leaves of the recursive tree.

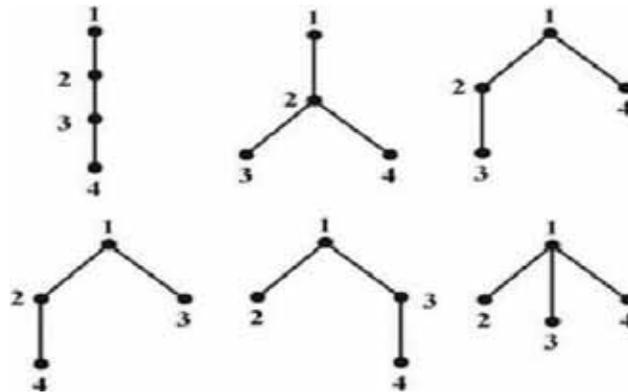


Fig. 1: All recursive trees of order 4

**Definition:**

Consider an urn containing a finite number of balls of different types (colors); say that the possible types are  $1, 2, \dots, p$ . Then the content of the urn at time  $n$  is described by the vector  $(Y_{n1}, \dots, Y_{np})$  where  $Y_{ni} \geq 0$  is the number of balls of type  $i$  in the urn we are further given, each type  $i$ , an actively  $\alpha_i \geq 0$ .

We start with  $Y_i$  balls of color  $i$  ( $i=1, \dots, p$ ). A draw consists of the following operations (Athreya and Karlin 1968)

- 1) select a ball random from the urn,
- 2) notice its color and return the ball to the urn, and
- 3) if  $c=i$ , add a random number  $R_{ij}$  of balls of color  $j$  ( $j=1, 2, \dots, p$ ) where the vector  $R_i = (R_{i1}, \dots, R_{ip})$  has the probability generating function  $h_i(s)$ . The stochastic process  $\{Y_n ; n = 0, 1, 2, \dots\}$  on the  $p$  dimensional integer lattice is called a *generalized Friedman's urn process*.

The sequence  $X_n$  from variables converges to random variable  $X$  almost surely (a.s), denoted by

$$X_n \xrightarrow{a.s} X \text{ if}$$

$$P\{w; X_n(w) \rightarrow X(w)\} = 1 \tag{3}$$

If  $\alpha_i = 1$  for each  $i$ , then

$$\frac{Y_n}{n} \xrightarrow{a.s} \lambda_1 v_1 \tag{4}$$

where  $v_1$  denotes the right eigenvector of  $A$  corresponding to the largest eigenvalue  $\lambda_1$ . Reader can be seen more details in Mahmoud and Smythe (1992).

In the next section by the generalized Friedman's urn and the transpose of matrix showed by Mahmoud and Smythe (1992), Smythe and Mahmoud (1995) the result of Meir and Moon is extended to a.s convergence.

**RESULTS AND DISCUSSION**

In our application, the balls in urn correspond to nodes in the recursive tree. We will need four colors B, R, G and W, as follows:

- Black denotes a leaf or external node of the tree;
- Red denotes a node of outdegree one;
- Green denotes a node of outdegree two;
- White denotes all other nodes.

Let  $B_n, R_n, G_n, W_n$  denote, respectively, the number of black, red, green and white balls after  $n$  drawing. The initial composition of the urn is given by  $B_0=1, R_0=G_0=W_0=0$ . The addition scheme for the balls may be represented as:

$$A = \begin{bmatrix} & B & R & G & W \\ B & 0 & 1 & 0 & 0 \\ R & 1 & -1 & 1 & 0 \\ G & 1 & 0 & -1 & 1 \\ W & 1 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

The reasoning behind this is straightforward: if a node outdegree  $k = 0$ ( black ),  $1$ ( red ), or  $2$ ( green ) is selected as the parent, it becomes a node of outdegree  $k+1$ ; if  $k = 0$ , no new leaves appear but if  $k = 1$  or  $2$ , the nodes of outdegree  $k$  decrease by one and one new leaf( black ) is added. Finally, selection of a node of outdegree  $>2$ ( white ) simply adds a new leaf ( black ). After  $n$  draws, the urn contains  $n + 1$  balls. Thus  $B_n + R_n + G_n + W_n = n + 1$ , i.e., the urn composition after  $n$  draws corresponds to a random recursive tree of order  $n+1$ .

Thus in method of Mahmoud and Smythe (1992) the space of nodes is divided corresponding following:

- 1) nodes with outdegree 0,1,2,
- 2) nodes with outdegree  $>2$ . We consider the following matrix:

$$A^T = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6}$$

We can assume that in the vector  $v_1$ , the right eigenvector of  $A^T$ ,  $v_{11} = \frac{1}{2}$  [ See the (Janson) , for this selection].

The urn evolves according to a Markov process ( Janson). At each time  $n \geq 1$ , one of the balls in the urn is drawn at random such that the probability of drawing a particular ball is drawn uniformly at random.

**Theorem:**

As  $n \rightarrow \infty$ ,

$$\frac{Y_{ni}}{n} \xrightarrow{a.s} \left(\frac{1}{2}\right)^{i+1}, \quad i = 0, 1, 2. \tag{7}$$

**Proof:**

In first, we calculate the eigenvalues of transpose matrix of Mahmoud and Smythe (1992). Thus

$$\det \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -1-\lambda & 1 & 0 \\ 1 & 0 & -1-\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda_1 = 1, \lambda_2, \lambda_3 = -1.$$

Since  $A^T v_1 = \lambda_1 v_1$ ,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \begin{cases} v_{11} + v_{12} + v_{13} = v_{10}, \\ v_{10} = 2v_{11} \\ v_{11} = 2v_{12} \\ v_{12} = v_{13}. \end{cases}$$

We see that  $v_{11} = v_{10} / 2$  and  $v_{12} = v_{11} / 2$ . In the other hand,  $v_{11}$  and  $v_{12}$  are half of  $v_{10}$  and  $v_{11}$ , respectively.

Since we do not take node of outdegree  $>2$  in account so we do not consider  $v_{12} = v_{13}$  in this case. Since  $\lambda_1 = 1$  is simple by note the Perron-frobenius theory (Karlin (1969),  $v_1$  may be chosen non-negative. In hence

$v_1 = (\frac{1}{2}, (\frac{1}{2})^2, (\frac{1}{2})^3, (\frac{1}{2})^3)$ , i.e,  $v_i = (\frac{1}{2})^{i+1}$  for  $i=0,1,2$ . So, the urn composition after  $n$  draws

corresponds to a random recursive tree of order  $n + 1$ , thus from (4),

$$\frac{Y_{n,i}}{n} \xrightarrow{a.s} \left(\frac{1}{2}\right)^{i+1}. \tag{8}$$

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