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On a New Numerical Scheme for the Solution of Initial Value Problems (Ivps) In Ordinary Differential Equation

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Abstract: In this paper we present the development, analysis and implementation of new algorithms which are particularly well suited for initial value problems in ordinary differential equation having oscillatory or exponential solutions.

Key words:

INTRODUCTION

Over the years, large number of methods suitable for solving sets of ordinary differential equations (ODES) have been proposed. Generally the efficiency of any of the methods depends on the method's stability and certain accuracy properties. The accuracy properties of different methods are usually compared by considering the order of convergence as well as the truncation error coefficient of the various methods (C.F Tischer 1984).

The sources of motivation for this work are those of Fatunla (1979, 1978a) and Ibijola (1997), Fatunla proposed a numerical integration scheme which is particularly well suited to solve initial value problem having oscillatory or exponential solutions. His method was based on the local representation of the theoretical solution y(x) to the Initial Value Problem of the form $y^1 = F(x,y)$, $y(a) = \eta$ in the interval (x_n, x_n+_1) by interpolating function $F(x) = a_0 + a_1 x + b\,real e^{(\rho x+\mu)}$, where a_0, a_1 and b are real undetermined coefficients while ρ and μ are complex parameters.

Ibijola in (1997) also proposed a numerical integration scheme suited for initial value problems of the form $y^1 = f(x,y)y(a) = \eta$ in the interval $[x_n, x_{n+1}]$ by a polynomial interpolating function $F(x) = a_0 + a_1 x + a_2 x^2 + b(real e^{\rho x+\mu})$. Where a_0, a_1, a_2,and b are real undetermined coefficients while ρ and μ are complex parameters.

In this work, we considered the initial problem $y^1 = f(x,y)y(a) = \eta$ and we develop an algorithm of order six which can effectively cope with the IVPs with oscillatory or exponential solution. Numerical experiments were carried out and the results compared with the existing results in particular with that of Fatunla (1976) which is order four and that of Ibijola (1997) which is of order five.

(2.) The Basic Interpolant:

Let us assume that the theoretical solution y(x) to the initial value problem

$$y^1 = f(x,y) \qquad y(a) = \eta \tag{1.1}$$

can be locally represented in the interval $(x_t\ x_{t+1})$, $t \geq 0$ by the polynomial interpolating function

$$F(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + b(real e^{\rho x+\mu}) \tag{1.2}$$

where a_0, a_1, a_2, a_3 and b are real undetermined coefficients and ρ, μ are complex parameters, if we put

$$\rho = \rho_1 + i\rho_2 \tag{1.3}$$

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$\mu = i\sigma, i^2 = -1$ in 1.2 we obtain the following interpolating function

$$F(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + b\ell^{\rho_1 x}\cos(\rho_2 x + \sigma) \tag{1.4}$$

We shall go further to define the function R(x) and $\Theta(x)$ as follow.

$$R(x) = \ell^{\rho_1 x} \tag{1.5}$$

and $\theta(x) = \rho_2 x + \sigma$ (1.6)

From (1.4), (1.5) and (1.6) we obtain

$$F(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + bR(x)\cos\theta(x) \tag{1.7}$$

we shall assume that y_t is a numerical estimate to the theoretical solution $y(x_t)$ and that $f_t = f(x_t, y_t)$. we define mesh points as follows

$x_t = a + th$

t = 0, 1, 2, 3 ------- (1.8)

We then move forward to impose the following constraints on the interpolating function (1.7)
(a) That the interpolating function must coincide with the theoretical solution at $x=x_t$ and $x=x_{t+1}$. in other words we required that

$$F(x_t) = a_o + a_1 x_t + a_2 x_t^2 + a_3 x_t^3 + bR(x_t)\cos\theta(x_t) \tag{1.9}$$

i.e $F(x_t)=y(x_t)$ and

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + a_2 x_{t+1}^2 + a_3 x_{t+1}^3 + bR(x_{t+1})\cos\theta(x_{t+1}) \tag{1.10}$$

Which implies that $F(X_{t+1}) = y(X_{t+1})$

(b) we also require that the first, second, third and fourth derivatives with respect to x of the interpolating function respectively coincide with the differential equation as well as its first, second, third and fourth derivatives with respect to x at $x = x_t$. In other words we require that

$$F^1(X_t) = f_t$$
$$F^2(X_t) = f_t^{(1)}$$
and
$$F^3(X_t) = f_t^{(2)}$$
$$F^4(X_t) = f_t^{(3)}$$

This implies that

$$F^i(x_t) = f_t = a_1 + 2a_2 x_t 3a_3 x_t^2 + bR(x_t)[\rho_1\cos\theta(x_t) - \rho_2\sin\theta(x_t)] \quad 1.11$$

$$f_t = a_1 + 2a_2 x_t + 3a_3 x_t^2 + bR_t\left[\rho_1\cos\theta_t - \rho_2\sin\theta_t\right] \quad 1.11a$$

$$F^{ii}(x_t) = 2a_2 + 6a_3 x_t + bR(x_t)\left[\left(\rho_1^2 - \rho_2^2\right)\cos\theta(x_t) - 2\rho_1\rho_2\sin\theta(x_t)\right]$$

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$$f_t^{(1)} = 2a_2 + 6a_3x_t + bR_t\left[\left(\rho_1^2 - \rho_2^2\right)\cos\theta_t - 2\rho_1\rho_2\sin\theta_t\right] \quad 1.12a$$

$$F^{iii}(x_t) = 6a_3 + bR(x_t)\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta(x_t) + (\rho_{12}^3 - 3\rho_1^2\rho_2)\sin\theta(x_t)\right] \quad 1.13$$

$$f_t^{(2)} = 6a_3 + bR_t\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_{12}^3 - 3\rho_1^2\rho_2)\sin\theta_t\right] \quad 1.13a$$

$$F^{iv}(x_t) = bR(x_t)\left[\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta(x_t) + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho 2\right)\sin\theta(x_t)\right] \quad 1.14$$

$$f_t^{(3)} = bR_t\left[\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta(t) + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho 2\right)\sin\theta(t)\right] \quad 1.14a$$

Note that $f_t = F^1(x_t), f_t^{(1)} = F^{11}(x_t), F^{111}(x_t), f_t^{(3)} = F^{Iv}(x_t)$

From equation 1.14a

$$b = \frac{f_t^{(3)}}{R_t\left[\left(\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho 2\right)\sin\theta_t\right)\right]}$$

From equation 1.13a

$$f_t^{(2)} = 6a_3 + bR_t\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]$$

$$6a_3 = f_t^{(2)} - bR_t\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]$$

$$a_3 = \frac{1}{6}\left[f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_{12}^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right] \quad 1.16$$

From equation 1.1a

$$f_t^{(1)} = 2a_2 + 6a_3x_t + bR_t\left[\left(\rho_1^2 - \rho_2^2\right)\cos\theta_t - 2\rho_1\rho_2\sin\theta_t\right], note \qquad x_t = (a + th)$$

$$2a_2 = f_t^{(1)} + 6a_3x_t = bR_t\left[\left(\rho_1^2 - \rho_2^2\right)\cos\theta_t - 2\rho_1\rho_2\sin\theta_t\right]$$

$$a_2 = \frac{1}{2}\left\{\begin{array}{l} f_t^{(1)} - \left[f_t^{(2)} - \dfrac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho^3{}_2 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right](a + th) \\ -\left[\dfrac{\left[\left(\rho_1^2 - \rho_2^2\right)\cos\theta_t - 2\rho_1\rho_2\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right] \end{array}\right\} \quad 1.17$$

From equation 1.1a

$$f_t = a_1 + 2a_a x_t + 3a_3x^2{}_t + bR_t\left[\rho_1\cos\theta_t - \rho_2\sin\theta_t\right]$$

$x_t = (a + th)$

$$x_t^2 = (a+th)^2 = a^2 + 2ath + t^2h^2$$

$$a_1 = f_t - 2a_a x_t - 3a_3 x^2{}_t = bR_t\left[\rho_1\cos\theta_t - \rho_2\sin\theta_t\right]$$

$$a_1 = f_t\left[f_t^{(1)} - \left[f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^2 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right](a+th)\right]$$

$$-\frac{1}{2}\left[f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right](a+th)^2$$

$$-\left[\frac{\left[\rho_1\cos\theta_t - \rho_2\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right] \qquad 1.18$$

$$y_{t+1} - y_t = a_1h + a_2(x_{t+1}^2 - x_t^2) + a_3(x_{t+1}^3 - x_t^3) + bR_t\left[\ell^{\rho_1 h}\left(\cos\theta_t\cos\rho_2 h - \sin\theta_t\sin\rho_2 h\right) - \cos\theta_t\right] \qquad 1.19$$

$$X_t = a + th, x^2t = (a+th)^2 \Rightarrow x_{t+1}^2 - x_t^2 = (a+(t+1)h)^2 - (a+th)^2 = 2ah + (1+2t)h^2 \qquad 1.20$$

$$X_{t+1}^2 = (a+(t+1)h)^2$$

$$X_t^3 = (a+th)^3, x_{t+1}^3 = (a+(t+1)h)^3 \Rightarrow x_{t+1}^3 - x_t^3 = (a+(t+1)h)^3(a+th)^3 = 3a^2h + ah^2$$
$$(3+6t) + h^2(3t^2+3t+1) \qquad 1.21$$

We can now obtain the new scheme using (1.19), putting the values b, a_1, a_2, a_3, in (1.19) also substituting (1.20) & (ii) in (1.19);
we have:
From equation (1.19)

$$y_{t+1} - y_t = a_1h + a_2(x_{t+1}^2 - x_t^2) + a_3(x_{t+1}^3 - x_t^3) + bR_t\left[\ell^{\rho_1 h}\left(\cos\theta_t\cos\rho_2 h - \sin\theta_t\sin\rho_2 h\right) - \cos\theta_t\right]$$

$$y_t\left\{\begin{array}{l} f_t - \left[f_t^{(1)} - \left(f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right)\right](a+th) \\ -\frac{1}{2}\left[f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right](a+th)^2 \\ -\left[\frac{\left[\rho_1\cos\theta_t - \rho_2\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right] \end{array}\right\}h$$

$$+\frac{1}{2}\left\{f_t^{(1)} - \left[f_t^{(2)} - \frac{\left[\left(\rho_1^3 - 3\rho_1\rho_2^2\right)\cos\theta_t + (\rho_2^3 - 3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4 + \rho_2^4 - 6\rho_1^2\rho_2^2\right)\cos\theta_t + \left(4\rho_1\rho_2^3 - 4\rho_1^3\rho_2\right)\sin\theta_t}\right](a+th)\right\}$$

$$-\left[\frac{\left[\left(\rho_1^2-\rho_2^2\right)\cos\theta_t-2\rho_1\rho_2\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4+\rho_2^4-6\rho_1^2\rho_2^2\right)\cos\theta_t+\left(4\rho_1\rho_2^3-4\rho_1^3\rho_2\right)\sin\theta_t}\right)\left(2ah+(1+2t)h^2\right)$$

$$+\frac{1}{6}\left\{\begin{array}{l}\left[f_t^{(2)}-\frac{\left[\left(\rho_1^3-3\rho_1\rho_2^2\right)\cos\theta_t+(\rho_{12}^3-3\rho_1^2\rho_2)\sin\theta_t\right]f_t^{(3)}}{\left(\rho_1^4+\rho_2^4-6\rho_1^2\rho_2^2\right)\cos\theta_t+\left(4\rho_1\rho_2^3-4\rho_1^3\rho_2\right)\sin\theta_t}\right]\\ \left(3a^2h+ah^2(3+6t)+h^3(3t^2+3t+1\right)\end{array}\right\}$$

$$+\left[\frac{\left[\ell^{\rho_1 h}\left(\cos\theta_t\cos\rho_2 h-\sin\theta_t\sin\rho_2 h\right)-\cos\theta_t\right]f_t^{(3)}}{\left(\rho_1^4+\rho_2^4-6\rho_1^2\rho_2^2\right)\cos\theta_t+\left(4\rho_1\rho_2^3-4\rho_1^3\rho_2\right)\sin\theta_t}\right] \qquad 1.22$$

Equation (1.22) is the new scheme

(3) Applications and Numerical Experiment:

Fatunla (1978a) and Roberton (1976) made it clear that mathematical formulation of physical situations in chemical kinetics population models, mechanical oscillations, planetary motion, electrical networks, nuclear reactor control, turnel switching problems, reversible enzyme kinetics often lead to initial value problems (IVP) of the form $y^1 = f(x,y), y(a)=\eta$ whose solutions are exponential or oscillatory. The new scheme is capable of solving problems arising from these mentioned situations leading to (IVPs). To obtain the numerical solution y_{t+1} at $X = X_{t+1}$ the function $f(x,y)$ and its higher derivatives are evaluated at $x = x_t$ the value obtained are used in the simultaneous equations. These equations are solved by Newton's iteration method to obtained the parameters $\rho_1, \rho_2 and \theta_t$. The numerical value of these parameters are then used in the scheme to generate y_{t+1}

Example 1:

Let us consider initial value problem $y^1 = y, y(0)=1$ (Lambert 1973a, Fatunla 1988), in the interval $0 \le x \le 1$.

The parameters ρ_1 ρ_2 and θ_i were obtained with $\rho_1 = 1.5000023$, 0.6999988, $\theta_t =$ 0.0085289.

The numerical experiments are shown below, the results show that our scheme compared favourably with that of Fatunal (1976) and Ibijola (1997).

		FATUNLA 1976 h =0.1		IBIJOLA h = 0.1			OGUNRINDE 2009 h=0.1		
t		y_t	$10^{6*}T_{t+1}$	y_t	$^*T_{t+1}$	$Y(x_t)$	Y_t	T_{t+1}	$Y(x_t)$
0	0.00	1.000000	0.00000	.1000000D+01	.0000000D+00	.1000000D+01	1.000000	1.000000	0.0000000
1	0.10	1.1051709	0.60450	.1105164D+01	.6867420D-05	.1105171D+01	1.1051710	1.0051710	0.0999997
2	0.20	1.2214028	0.448289	.1221388D+01	.1508643D-04	.1221403D+01	1.2214028	1.0490481	0.1723547
3	0.30	1.3498588	0.38781	.1349834D+01	2507501D-04	.1349859D+01	1.3498589	1.1266546	0.2232043
4	0.40	1.4918247	0.36697	.1491788D+01	.3688326D-04	.1491825D+01	1.4198247	1.2362691	0.2555556
5	0.50	1.6487213	0.32105	.1648670D+01	.5082665D-04	.1648721D+01	1.6487212	1.3784846	0.2702366
6	0.60	1.8221188	0.02256	.1822051D+01	6754957D-04	.1822119D+01	1.8221189	1.5557216	0.2663972
7	0.70	2.0137527	0.85937	.201366 D+01	.8699242D-04	.2013753D+01	2.0237527	1.7719986	0.2417541
8	0.80	2.2255409	0.34473	.2225431D+01	1098384D-03	.222541 D+01	2.2255409	2.0328526	0.1296882
9	0.90	2.4596031	0.30697	.2459660 D+01	.13688352D-03	.2459603D+01	2.4596033	2.3453450	0.1142583
10	1.00	2.7182818	0.03030	.2718114 D+01	.1677045D-03	.2718282D+01	2.7182817	2.7181075	0.0001743

Example 2:

Consider the scalar initial value problem y^1=-2xy+4x, y(0)=3 in the interval 0≤ x≤. The exact solution is given by y(x) = $\ell^{-x^2}+2$. The numerical experiment was carried out and the parameter $\rho_1 = 3.2710082$, $\rho_2 = 3.500012$, $\theta_t = 1.410003$ were obtained to seven decimal places. The results are shown below.

t	h	$Y(x_t)$	FATUNLA (1976)	IBIJOLA (1997)	OGUNRINDE (2009)
0	0.00	3.0000000	3.0000000	3.0000000	3.0000000
1	0.10	2.9900472	2.9900471	2.9900474	2.99805689
2	0.20	2.9607869	2.9607868	2.9607887	2.97623372
3	0.30	2.9139286	2.9139287	2.9139309	2.93558693
4	0.40	2.8521414	2.8521414	2.8521433	2.87826848
5	0.50	2.7787983	2.7787986	2.7788000	2.80733442
6	0.60	2.6976833	2.6976844	2.6976757	2.72647023
7	0.70	2.6126237	2.6126247	2.6126261	2.63967109
8	0.80	2.5272812	2.5272810	2.5272923	2.55091238
9	0.90	2.4448478	2.4448468	2.4448576	2.46385169
10	1.00	2.3678764	2.3678784	2.3678789	2.36704040

The results obtained confirmed that our scheme is capable of solving initial value problems with exponential solutions.

Conclusion:

We have developed a method which is consistent and convergent, a method which can cope with problem having oscillatory or exponential solutions. The scheme compared favourably with the existing, standard scheme especially that of Fatunla (1976) and Ibijola (1997). We must point out that Fatunla (1976) and Ibijola (1997) have derivation advantage over our new scheme in that fewer derivatives are needed in the implementation. However our method enjoys higher order advantage over that of Fatunla (1976) which is of order four and Ibijola (1997) of order five while our own is of order six.

Computational results indicate that the new scheme is accurate and efficient for oscillatory system and for initial value problem with exponential solutions. The fact that the new scheme requires in some cases very few functions evaluation confirms that the method is of great importance.

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