Australian Journal of Basic and Applied Sciences, 4(11): 5508-5515, 2010
ISSN 1991-8178

Development of the Sinc Method for Nonlinear Integro-Differential Eequations

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Abstract: We develop a numerical procedure for solving a class of nonlinear integro-differential equations of Fredholm type, using the globally defined Sinc basis functions. Properties of the Sinc procedure are utilized to reduce the computation of the integro-differential equations to system of nonlinear equations with unknown coefficients. We used three numerical examples to illustrate the accuracy and implementation of our method.

Key words: Fredholm; Integro-differential; Nonlinear; Sinc method.

INTRODUCTION

We consider the following nonlinear Fredholm integro-differential equation of the form:

$$u'(x) = f(x) + \lambda \int_{\Gamma} K(x,t) g(t,u(t)) dt, \qquad x \in \Gamma = [a,b], \tag{1}$$

where the functions $f(x)$ and the kernel $K(x,t)$ are known and $u(x)$ is the solution to be determined (Atkinson, 1997; Delves and Mohamed, 1985), and also $g(t,u(t))$ is nonlinear in u(t).

Integro-differential equations have strong physical background and also, have many practical applications in scientific fields such as population and polymerrheology (Linz, 1985; Abdul Jerri, 1999). Nonlinear phenomena, that appear in many applications in scientific fields can be modeled by nonlinear integro-differential equations. In recent years, numerous methods have been applied for solving integral and integro-differential equations such as variational iteration method (Wang and He, 2007), rationalized Haar functions (Ordokhani and Razzaghi, 2008), CAS wavelet method (Danfu and Xufeng, 2007), Adomian method (El-Kalla, 2008), Homotopy perturbation method (Yusufoˇglu, 2009) and Taylor polynomial approach (Kurt and Sezer, 2008).

Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied physics and engineering (Stenger, 1993). The Sinc-Collocation overlapping method is developed for two-point boundary-value problems for second-order ordinary differential equations in (Morlet, Lybeck and Bowers, 1999). In (Rashidinia and Zarebnia, 2005), we used a Sinc-collocation procedure for numerical solution of linear Fredholm integral equations of the second kind. In this paper a global approximation for the solution of the equation (1) using the Sinc functions is developed. Our method consists of reducing the solution of (1) to a set of algebraic equations. The properties of Sinc function are then utilized to evaluate the unknown coefficients.

The outline of the paper is as follows. In section 2 we review some of the main properties of Sinc function that are necessary for the formulation of the discrete system. In section 3, we illustrate how the Sinc method may be used to replace Eq. (1) by an explicit system of nonlinear algebraic equations. In section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

2 Sinc Function Properties and Interpolation:

Sinc function properties are discussed thoroughly in (Stenger, 1993). In this section an overview of the basic formulation of the Sinc function required for our subsequent development is presented.
The Sinc function is defined on the whole real line by

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$$Sinc(x)=\begin{cases}\dfrac{\sin(\pi x)}{\pi x}, & x\neq 0\\ 1, & x=0;\end{cases} \tag{2}$$

For any $h>0$, the translated Sinc functions with evenly spaced nodes are given as follows:

$$S(j,h)(x)=Sinc(\frac{x-jh}{h}),\qquad j=0,\pm1,\pm2,... \tag{3}$$

which are called the jth Sinc functions. The Sinc function form for the interpolating point $x_k = kh$ is given by

$$S(j,h)(kh)=\delta_{jk}^{(0)}=\begin{cases}1, & k=j;\\ 0, & k\neq j;\end{cases} \tag{4}$$

Let

$$\delta_{kj}^{(-1)}=\frac{1}{2}+\int_0^{k-j}\frac{\sin(\pi t)}{\pi t}dt, \tag{5}$$

then define a matrix $I^{(-1)}=[\delta_{kj}^{(-1)}]$ whose $(k,j)th$ entry is given by $\delta_{kj}^{(-1)}$. If *u* is defined on the real line, then for $h>0$ the series

$$C(u,h)(x)=\sum_{j=-\infty}^{+\infty}u(jh)Sinc(\frac{x-jh}{h}), \tag{6}$$

is called the Whittaker cardinal expansion of *u*, whenever this series converges. For further explanation of the procedure, we consider the following definitions and theorems.

Definition 1:

Let D be a simply-connected domain in the complex plane $(z=x+iy)$ having boundary ∂D. Let *a* and *b* denote two distinct points of ∂D and ϕ denote a conformal map of *D* onto D_d where D_d denote the region $\{w\in\mathbb{C}:|Iw|<d,\ d>0\}$ such that $\phi(a)=-\infty$ and $\phi(b)=+\infty$. Let $\Psi=\phi^{-1}$ denote the inverse map, and let Γ be defined by $\Gamma=\{z\in\mathbb{C}:z=\psi(u),u\in R\}$. Given ϕ, Ψ and a positive number *h*, let us set $z_k=z_k(h)=\psi(kh),\ \ k=0,\pm1,\pm2,\ldots.$

Definition 2:

Let $L_\alpha(D)$ be the set of all analytic functions, for which there exists a constant, C, such that

$$\left|F(z)\right|\le C\frac{\left|\rho(z)\right|^{\alpha}}{\left[1+\left|\rho(z)\right|\right]^{2\alpha}}\qquad z\in D,\quad 0<\alpha\le1, \tag{7}$$

where $\rho(z)=e^{\phi(z)}$.

Theorem 1:

Let $\frac{u}{\phi'} \in L_\alpha(D)$, let N be a positive integer and let h be selected by the formula

$$h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}, \tag{8}$$

then there exist a positive constant c_1, independent of N, such that

$$\left| \int_\Gamma u(z)dz - h\sum_{k=-N}^{N} \frac{u(z_k)}{\phi'(z_k)} \right| \le c_1 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{9}$$

Theorem 2:

Let $\frac{u}{\phi'} \in L_\alpha(D)$, with α >0, and d>0, let $\delta_{kj}^{(-1)}$ be defined as in (5), and let $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$. Then there exists a constant, c_2 which is independent of N, such that

$$\left| \int_a^{z_k} u(t)dt - h\sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(z_j)}{\phi'(z_j)} \right| \le c_2 e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \tag{10}$$

Solution of Fredholm-Hammerstein Integro-Differential Equation:

We consider the nonlinear Fredholm-Hammerstein integro-differential equation of the form

$$u'(x) = f(x) + \lambda \int_\Gamma K(x,t)g(t,u(t))dt, \qquad x \in \Gamma = [a,b], \tag{11}$$

with initial condition $u(a) = u_a$.

In Eq. (11) f and the kernel K are continuous functions, and also $g(t,u(t))$ is nonlinear in u. For convenience, consider

$Q(t) = g(t, u(t))$.

Let $u(x)$ be the exact solution of the integral equation (11) and $u(x) \in L_\alpha(D)$. By considering Eq. (11) and integrating (11) from a to x, we get:

$$u(x) = \int_a^x \{\lambda \int_\Gamma K(\xi,t)Q(t)dt + f(\xi)\}d\xi + u(a), \qquad x \in \Gamma = [a,b]. \tag{12}$$

We consider

$$\tilde{K}(\xi) = \int_\Gamma K(\xi,t)Q(t)dt. \tag{13}$$

Now, we let $\frac{\tilde{K}}{\phi'} \in L_\alpha(D)$ and $\frac{f}{\phi'} \in L_\alpha(D)$. By setting $x = x_k$, $k = -N,...,N$ and applying Theorem 2 for the right-hand side of (12), we have:

$$\int_a^{x_k} \tilde{K}(\xi)d\xi \simeq \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\tilde{K}(x_l)}{\phi'(x_l)}, \tag{14}$$

$$\int_a^{x_k} f(\xi)d\xi \simeq \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f(x_l)}{\phi'(x_l)}. \tag{15}$$

Thus we obtain:

$$u(x_k) \simeq \lambda h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\tilde{K}(x_l)}{\phi'(x_l)} + h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f(x_l)}{\phi'(x_l)} + u_a. \tag{16}$$

For the first term on the right-hand side of above relation, by considering relation (13) and using Theorem 1, we get:

$$h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\tilde{K}(x_l)}{\phi'(x_l)} \simeq h^2 \sum_{l=-N}^{N} \sum_{j=-N}^{N} \delta_{k,l}^{(-1)} \frac{K(x_l,t_j)}{\phi'(x_l)\phi'(t_j)} Q(t_j), \tag{17}$$

where $Q(t_j) = g(t_j, u(t_j))$. Having replaced the first term on the right-hand side of (16) with the equation (17), we obtain:

$$u_k - \lambda h^2 \sum_{l=-N}^{N} \sum_{j=-N}^{N} \delta_{k,l}^{(-1)} \frac{K_{l,j}}{\phi_l' \phi_j'} Q_j \simeq h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_1}{\phi_l'} + u_a, \tag{18}$$

where $u_k = u(x_k)$, $K_{l,j} = K(x_l, t_j)$, $Q_j = Q(t_j)$, $f_l = f(x_l)$ and

$$\phi(x) = \ln(\frac{x-a}{b-x}), \quad \phi(a) = -\infty, \quad \phi(b) = +\infty,$$

$$\phi'(x) = \frac{b-a}{(x-a)(b-x)}, \quad x_k = \psi(kh) = \phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}.$$

There are (2N+1) unknowns $u_j, j = -N, -N+1, \cdots, N-1, N$ to be determined in (18). In order to determine these (2N+1) unknowns, we rewrite these system which is the nonlinear system of equations in matrix form. Corresponding to a given function u defined on Γ we use the notation $D(u) = diag(u(x_{-N}, \cdots, u(x_N))$. We set $I^{(-1)} = [\delta_{kj}^{(-1)}]$, where $\delta_{kj}^{(-1)}$ denotes the $(k, j)th$ element of the matrix $I^{(-1)}$. Now, we can simplify the system (18) in the matrix form

$$U - A\tilde{Q} = \Phi, \tag{19}$$

where

$$A = \lambda h^2 (I^{(-1)} D(\frac{1}{\phi'}))(\hat{K} D(\frac{1}{\phi'})),$$

$$\hat{K} = [K(x_i, t_j)], \quad i, j = -N, \ldots, N,$$

$$\tilde{Q} = [Q_{-N}, Q_{-N+1}, \ldots, Q_{N-1}, Q_N]^T,$$

$$\Phi = [h \sum_{l=-N}^{N} \delta_{-N,l}^{(-1)} (\frac{f_l}{\phi_l'}) + u_a, \cdots, h \sum_{l=-N}^{N} \delta_{N,l}^{(-1)} (\frac{f_l}{\phi_l'}) + u_a]^T,$$

$$U = [u_{-N}, \cdots, u_N]^T.$$

The above nonlinear system consists of (2N+1) equation with (2N+1) unknown $\{u_j\}_{j=-N}^{N}$. Solving this nonlinear system by Newton`s method, we obtain unknown coefficients $u_j, j=-N,-N+1,\cdots,N.$ Having used the approximate solution $u_j, j=-N,-N+1,\cdots,N$, we employ a method similar to the Nystrom's idea for the Ferdholm- Hammerstein integro-differential equation, i.e., we use

$$u_N(x)=\lambda h^2\sum_{l=-N}^{N}\sum_{j=-N}^{N}\frac{K(\xi_l)}{\phi'(\xi_j)\phi'(t_l)}\Omega_{h,j}(x)Q_l+h\sum_{l=-N}^{N}\frac{f(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)+u_a,$$

Where

$$\Omega_{h,l}(x)=\frac{1}{2}+\int_a^x S(l,h)o\phi(t)dt.$$

Each Newton iteration step involves evaluation of the vector $F^{(l)}$ the Jacobian matrix $J^{(l)}$ and $\Delta U^{(l)}$. Whenever the distance between two iteration is less than a given tolerance, ε, then the algorithm is to stop.

$$|U^{(l+1)}-U^{(l)}\|\le\varepsilon.$$

Algorithm:

- initialize: $U=U^{(0)}$
- for $l=0,1,2,\dots$
- $F^{(l)}=U|^{(l)}-A\tilde{Q}|^{(l)}-\Phi$
- if $|F^{(l)}\|$ is small enough, stop
- compute $J^{(l)}$
- solve $J^{(l)}\Delta U^{(l)}=-F(U^{(l)})$
- $U^{(l+1)}=U^{(l)}+\Delta U^{(l)}$
- end

4 Numerical examples:

We consider the following examples to compare our computed results and justify the accuracy and efficiency of our method. The examples have been solved by presented method with different values of $N,\alpha=1$ and $d=\frac{\pi}{2}$ which yield $h=\pi(\frac{1}{N})^{\frac{1}{2}}$. Let $u(x_j)$ denote the exact solution of the given examples, and let $u_N(x_j)$ be the computed solutions by our method. Let $\Gamma=[a,b]$ and ϕ be a conformal map onto D. By exploiting of the definition 1, we have

$$D=\{z\in\mathbb{C}:|\arg(\frac{z-a}{b-z}|<d\le\frac{\pi}{2}\}, \tag{20}$$

$$\phi(z)=\ln(\frac{z-a}{b-z}).$$

The error is reported on set of the Sinc grid points

$$S=\{x_{-N},\cdots,x_0,\cdots,x_N\},$$

$$x_k = \frac{a + be^{kh}}{1 + e^{kh}}, \qquad k = -N, \cdots, N. \tag{21}$$

The maximum absolute error on the grid points Sinc is defined as

$$\| E_s(h) \| = \max_{-N \le j \le N} | u(x_j) - u_N(x_j) |. \tag{22}$$

The maximum absolute errors in numerical results are tabulated in tables 1 – 3.

Example 1:

Consider the following nonlinear Fredholm integro-differential equation with exact solution $u(x) = x$

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 x u^2(t)dt, \qquad 0 \le x \le 1,$$

$$u(0) = 0,$$

We applied the Sinc function approach and solved example 1. Maximum absolute errors in numerical solution of example 1 are tabulated in Table 1. These results show the efficiency and applicability of our method. The exact and approximate solutions of example 1 are shown in Fig.1 for *N*=1 and *N*=4. For large values of *N* the approximate solution is indistinguishable(for the given scale) from the exact solution.

Table 1: Results for Example 1.

N	h	$\| E_s(h) \|_\infty$
5	1.404963	1.52165×10^{-3}
10	0.993459	9.69864×10^{-5}
20	0.702481	1.75941×10^{-6}
30	0.573574	7.86157×10^{-8}
40	0.496729	5.65671×10^{-9}
50	0.444288	5.52471×10^{-10}
60	0.405578	2.54344×10^{-11}

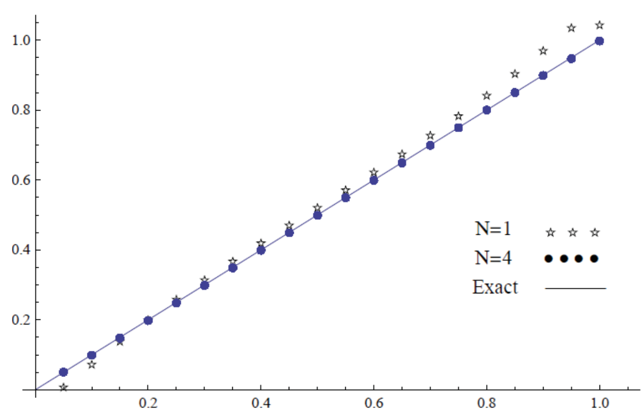

Fig. 1: Exact and approximate solutions for Example 1, (*N*=1,4).

Example 2: We consider the integro-differential equation

$$u'(x) = e^x - \frac{1}{5}e^{-x^2}(e^5 - 1) + \int_0^1 e^{2t - x^2} u^3(t)dt,$$

$$u(0) = 1,$$

with exact solution $u(x) = e^x$.

We solved the example 2 by our method for different values of N. The maximum absolute errors on the Sinc grids S are tabulated in Table 2. The exact and approximate solutions of example 2 are shown in Fig.2 for N=1 and N=4.

Table 2: Results for Example 2.

N	h	$\|E_s(h)\|_\infty$
5	1.404963	3.72499×10^{-3}
10	0.993459	2.46788×10^{-4}
20	0.702481	5.41244×10^{-6}
30	0.573574	2.51671×10^{-7}
40	0.496729	1.89372×10^{-8}
50	0.444288	1.85672×10^{-9}
60	0.405578	2.27677×10^{-10}

Example 3:
Consider the nonlinear Fredholm integro-differential equation

$$u'(x) = 1 - \frac{1}{2}(2 + e(x-1) - x) + \int_0^1 \frac{x - t^2}{2} e^{u(t)} dt, \qquad 0 \le x \le 1,$$

$$u(0) = 0,$$

with exact solution $u(x) = x$.

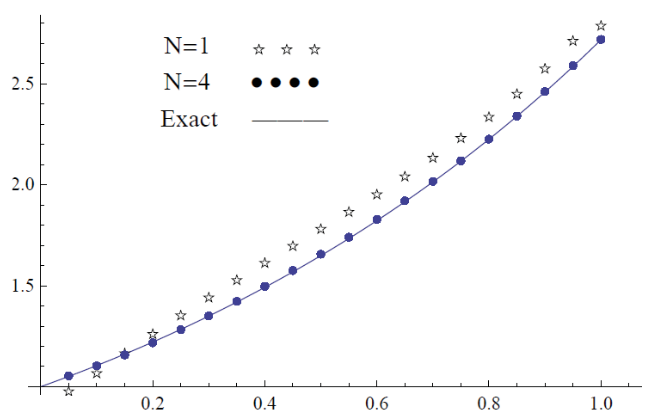

Fig. 2: Exact and approximate solutions for Example 2, (N=1,4).

The maximum absolute errors in computed solutions are tabulated in Table 3. These results show the accuracy and efficiency of our Sinc method. The exact and approximate solutions for example 3 are shown in Fig.3, including the approximations for N=1 and N=4, which are indistinguishable (on this scale) from the exact solution.

Table 3: Results for Example 3.

N	h	$\|E_s(h)\|_\infty$
5	1.404963	1.00069×10^{-3}
10	0.993459	6.06869×10^{-4}
20	0.702481	1.05499×10^{-6}
30	0.573574	4.59647×10^{-8}
40	0.496729	3.28830×10^{-9}
50	0.444288	3.17751×10^{-10}
60	0.405578	3.83886×10^{-11}

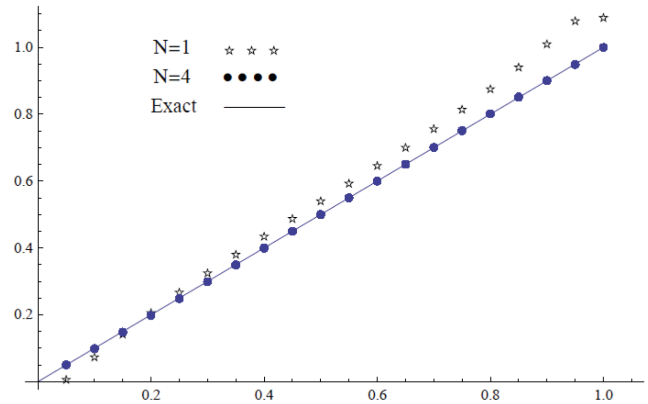

Fig. 3: Exact and approximate solutions for Example 3, (N=1,4).

Conclusion:

The Sinc method is used to solve the first-order Fredholm type integro-differential equation with initial condition. The numerical results show that the accuracy improves with increasing the N. The Tables 1–3 and Figures 1–3 indicate that as N increases the errors are decrease more rapidly, then for better results, using the larger N is recommended.

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