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Numerical Scheme for Fredholm Integral Equations Optimal Control Problems via Bernstein Polynomials

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Abstract: In this paper, we present a novel iterative method to approximate the solution to a class of optimal control problems governed by Fredholm integral equations. We are willing to construct a direct scheme based on the Bernstein polynomials and parameterization. The convergence of the method is also discussed in details and at the end, some numerical examples illustrate the efficiency and accuracy of the method.

Key words: Optimal control problem, Fredholm integral equations, Iterative methods, Approximate-Analytical solution, Numerical scheme.

INTRODUCTION

The advantages of optimal control theory and calculus variations are well established, during recent decades. The optimal control theory provides a systematic and direct approach to a large variety of control design problems including constrained optimization with interrelated manipulated variables (Stems *et al.* 2002).

Furthermore, optimal control theory along with optimization methods are presently employed for various applications in different fields, e.g., aerodynamics, meteorology, chemistry, nuclear magnetic resonance, acoustics, economical models, financial mathematics, pharmaceutical manufacturing, computational biology, and bioinformatics (Homescu *et al.* 2003; Zdenek *et al.* 2009).

In mathematical formulation of physical phenomena, integral equations are always encountered and have attracted much attention. Integral equations are appeared in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in biology, quantum mechanics, optimization, optimal control systems, mathematical economics, population genetics, medicine, fluid mechanics, steady state heat conduction, and radiative heat transfer problems (Abdou *et al.* 2003; Babolian *et al.* 2004; Huang *et al.* 1995; Jiang *et al.* 2004; Kythe *et al.* 2002; Liang *et al.* 2004; Maleknejad *et al.* 2004; Maleknejad *et al.* 2005; Maleknejad *et al.* 2005; Wang 2006; Yang 2000; Zhang 1987).

In this paper, we consider the numerical solution of a class of optimal control problems governed by integral equations, which is described by the following minimization problem:

$$Minimize\ \ J(x,u) = \int_0^1 f_0(t, x(t), u(t))dt, \tag{1}$$

subject to

$$x(t) = y(t) + (2)\int_a^b k(t, s, u(s))x(s)ds,\ \ a.e\ on\,[a,b]$$

where $f_0 \in C([a,b]) \times R \times R$ and $y(.) \in C^\infty([a,b])$ are given functions, $x(t), u(t) \in C^\infty([a,b]])$ are the trajectory and control functions, respectively, which to be determined and the given kernel function, $k(t,s,u(s))$ is smooth in $C^\infty([a,b]) \times C([a,b]) \times R$. Here, we assume that the problem (1)-(2) has a unique solution.

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Due to the lack of existing an appropriate numerical method for solving this kind of optimal control problems, the main purpose of this study is to present a direct numerical scheme for obtaining approximate solutions of the problem (1)-(2) by using parameterization and Bernstein polynomials.

2 An Approximate-Analytical Solution to FIE2:

In this section, following the work of Mandal (Mandal *et ai.* 2007), it is assumed that for a given *u(s)*, the Fredholm integral equation (2) with *k*, a continuous and square integrable function, has an unique solution. To find an appropriate solution of (2), *x(t)* is approximated in the Bernstein polynomial basis in *[a, b]* as

$$x(t) = \sum_{i=0}^{n} a_i B_{i,n}(t), \tag{3}$$

where $B_{i,n}(t)\,(i = 0,1,...,n)$ are Bernstein polynomials of degree *n* defined on *[a, b]* as

$$B_{i,n}(t) = \binom{n}{i} \frac{(t-a)^i (b-t)^{n-i}}{(b-a)^n}, \quad i = 0,1,2,...,n \tag{4}$$

and $a_i\ (i = 0,1,...,n)$ are unknown constants to be determined. Substituting (3) in (2), we obtain

$$\sum_{i=0}^{n} a_i B_{i,n}(t) = y(t) + \sum_{i=0}^{n} a_i \int_a^b k(t,s,u(s)) B_{i,n}(s), \quad a < t < b \tag{5}$$

Multiplying both side by $B_{i,n}(t)\,(i = 0,1,...,n)$ and integrating both sides with respect to *t* between *t* = 0 and *t* = 0, we obtain the linear system

$$\sum_{i=0}^{n} a_i c_{ij} = b_j, \quad j = 0,1,2,...,n, \tag{6}$$

where

$$c_{ij} = \int_a^b [B_{i,n}(t) - \int_a^b k(t,s,u(s)) B_{i,n}(s) ds] B_{j,n}(t) dt, \quad i,j = 0,1,...,n \tag{7}$$

and

$$b_j = \int_a^b y(t) B_{j,n}(t) dt. \tag{8}$$

The linear system (6) can be solved by any standard method to produce $a_i\ (i = 0,1,...,n)$. These a_1 's when substituted in (3) produce *x(t)* approximately.
Here we take *[a, b] = [0,1]*.

3 The Solution to the Optimal Control Problem:

Let *Q* be the subset of the product space $C^\infty([0,1]) \times C^\infty([0,1])$ contains all pairs $(x(.), u(.))$, which satisfy the equation (2). Also, let $Q_{m,n}$ be the subset of *Q* consisting of all pairs $(x_m(.), u_n(.))$, where $u_n(.)$ is a parameterized control function as the following polynomial

$$u_n(t) = \sum_{i=0}^{n} a_i t^i, \tag{9}$$

and $x_m(.)$ is the extracted solution of the integral equation (2), which is considered as a polynomial of degree at most m

$$x_m(t) = \sum_{j=0}^{m} e_j(a_0, a_1, ..., a_n)t^j. \tag{10}$$

Here, $e_j : R^n \to R, j = 0,1,...,m$ are continuous functions. Now, we consider the minimizing of J on $Q_{m,n}$ with $\{a_k\}_{k=0}^{n}$ as unknowns. This is obviously an optimization problem in $n + 1$ dimensional space

$$\{(a_0, a_1, ..., a_n) \in R^{n+1} : a_0 = u_n(0) = u_0, \sum_{k=0}^{n} a_k = u_n(1) = u_1\}$$

and $J(x_m, u_n)$ may be considered as a function $J(a_0, a_1, ..., a_n)$.

Suppose, $(x_m^*(.), u_n^*(.))$ be the solution of minimizing J on $Q_{m,n}, m = 1,2,...;\ n = 1,2,...$, then the polynomial form of $u_n^*(.), n = 1,2,...$ in (9) and the Eq. (10) allow us to apply the presented method (Section 2) for extracting polynomial solution of (10), which results in obtaining a sequence of trajectory functions $\{x_m^*(...)\}_{m=1}$ as Bernstein polynomials, and finally to achieve a minimizing sequence $\{(x_m^*(.), u_n^*(.)\}_{m,n}$.

Lemma 3.1:

If $\alpha_{m,n} = \inf_{Q_{m,n}} J$ for $m,n = 1,2,...$, then $\{\alpha_{m,n}\}_{m,n=1}^{\infty}$ is a convergent sequence.

Proof. See (S. Fard *et al.* 2010).

Theorem 3.2:

If $\lim_{m,n\to\infty} \alpha_{m,n} = \alpha$ then $\alpha = \inf_Q J$.

Proof. By Lemma 3.1, let $\{\alpha_{m,n}\}$ converges to namely $\hat{\alpha} \geq \alpha$. By contradiction, if $\hat{\alpha} > \alpha$, then $\varepsilon = \frac{\hat{\alpha} - \alpha}{2} > 0$. Hence, there exists $(x(.), u(.))$, such that

$$J(x(.), u(.)) < \alpha + \varepsilon = \frac{\hat{\alpha}}{2} + \frac{\alpha}{2}. \tag{11}$$

From the continuity of J, there is a $\delta > 0$ Where

$$|J(v(.), w(.)) - J(x(.), u(.))| < \varepsilon, \tag{12}$$

whenever

$$\|(v(.), w(.)) - (x(.), u(.))\|_\infty < \delta, \tag{13}$$

Here $\|.\|_\infty$ is a norm on the vector space $C^\infty([0,1]) \times C^\infty([0,1])$, which can be defined as follows:

$$\|(v(.),w(.))\|_\infty = \|v(.)\|_\infty + \|w(.)\|_\infty ,$$

where, the norm properties can be checked easily. On the other hand, the set of all polynomial pairs are dense in $C^\infty([0,1])\times C^\infty([0,1])$, so there is a pair of polynomials $p_m(t)$ of degree at most m and $q_n(t)$ of degree at most n such that,

$$\|(p_m(.),q_n(.))-(x(.),u(.))\|_\infty < \frac{\delta}{3}. \tag{14}$$

Whereas, the pair $(p_m(.),q_n(.))$ does not satisfy

$$(p_m(0),q_n(0))=(x_0,u_0),(p_m(1),q_n(1))=(x_1,u_1),$$

so, we have to define another polynomials

$$v_m(t)=p_m(t)+(x_0-p_m(0))(1-t)+(x_1-p_m(1))t,$$
$$w_n(t)=q_n(t)+(u_0-q_n(0))(1-t)+(u_1-q_n(1))t,$$

that satisfy $(v_m(0),w_n(0))=(x_0,u_0)$ and $(v_m(1),w_n(1))=(x_1,u_1)$ so $(v_m,w_n)\in Q_{m,n}$. From (15) for t = 0,1 we have

$$\|(p_m(0),q_n(0))-(x_0,u_0)\|_\infty < \frac{\delta}{3}, \|(p_m(1),q_n(1))-(x_1,u_1)\|_\infty < \frac{\delta}{3},$$

Now for $t\in[0,1]$ by definition $v_m(.)$ and $w_n(.)$ we have

$$\|(v_m(.),w_n(.))-(x(.),u(.))\|_\infty \le \|(p_m(t),q_n(t))-(x(t),u(t))\|_\infty +$$

$$\|(p_m(0),q_n(0))-(x_0,u_0)\|_\infty (1-t)+\|(p_m(1),q_n(1))-(x_1,u_1)\|_\infty t$$

$$< \frac{\delta}{3}+\frac{\delta}{3}+\frac{\delta}{3}=\delta.$$

Therefore

$$\|(v_m(.),w_n(.))-(x(.),u(.))\|_\infty < \delta,$$

and (13)-(14) imply that

$$|\,J(v_m(.),w_n(.))-J(x(.),u(.))\,|_\infty < \varepsilon,$$

and so from (12)

$$J(v_m(.),w_n(.)) < \frac{\alpha}{2}-\frac{\alpha}{2}+J(x(.),u(.)) < \hat{\alpha},$$

a contradiction appears with $(v_m(.),w_n(.))\in Q_{m,n}$, so $\hat{\alpha}=\alpha$.

Now, we can summarize the above results in a numerical algorithm for obtaining approximate optimal control of (1) subject to Eq. (2).

4 Numerical Experiments:

In order to validate the optimal control formulation and to test the proposed numerical solution procedures, we present results of numerical experiments with two test problems and the numerical calculations are all undertaken by MATLAB software.

Example 4.1: In the first example, we consider the following optimal control problem

$$Minimize\ \ J = \int_0^1 (x(t) - 0.8182 - 2.7273t^2)^2 + (u(t) - t^2)^2\, dt, \tag{15}$$

subject to Fredholm integral equation [3]

$$x(t) = y(t) + \int_0^1 (t^2 + u(s)) \times x(s) ds. \tag{16}$$

where $y(t) = t^2$.

The Exact optimal solutions of (15)-(16) are

$$x^*(t) = 0.8182 + 2.7273t^2, \textit{and} \quad u^*(t) = t^2.$$

with the optimal criterion

$$J = J(x^*(t), u^*(t)) = 0.$$

Using the above method, we have the numerical results obtained in Table 1. As can be seen from Table 1 and Figures 1-3, the rapid convergence of the scheme is issued when the integers *n* and *m* are increased until 2.

Table 1: The Approximate-Analytical results for Example 4.1

Iteration	*n*	*m*	*x(t)*	*u(t)*	*J(x(t), u(t))*
1	1	1	$0.3637+2.7274t$	$-0.4156+1.4081t$	0.0469
2	1	2	$0.3640+2.7279t$	$-0.4363+1.4891t-0.0653t^2$	0.0618
3	2	2	$0.8182+2.7273t^2$	t^2	0

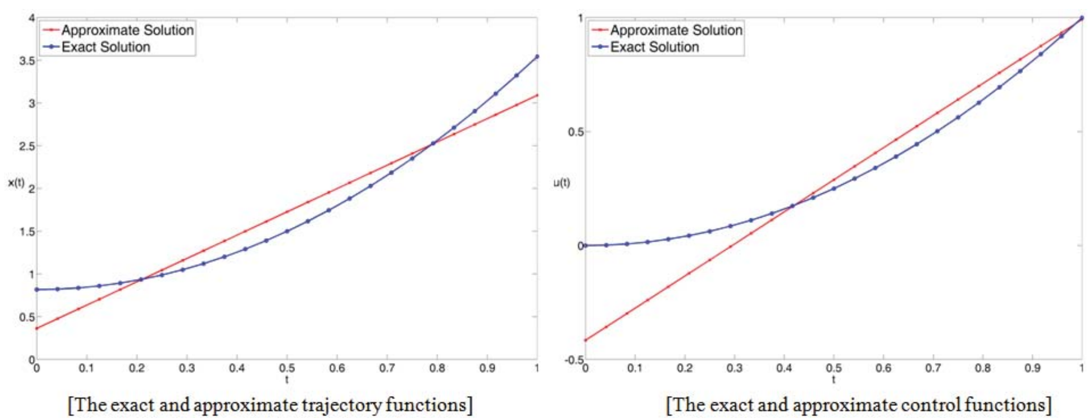


Fig. 1: Trajectory and control functions for Example 4.1, *n*=1, *m*=1.

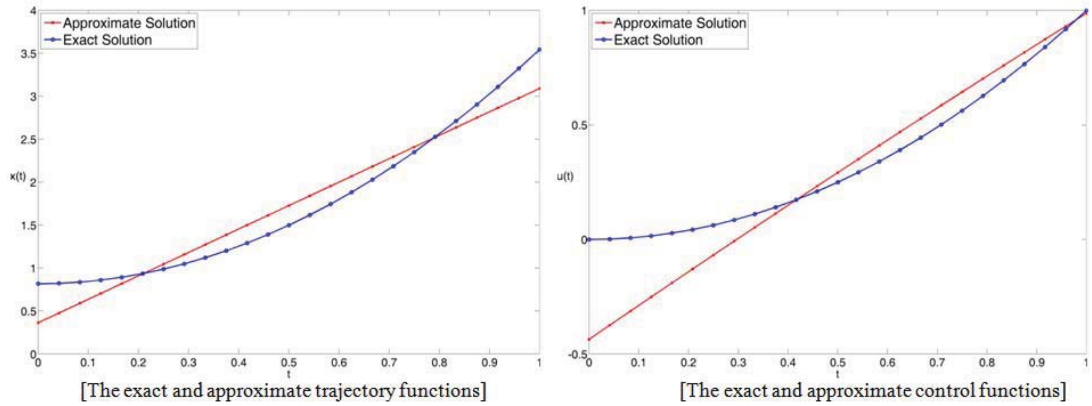


Fig. 2: Trajectory and control functions for Example 4.1, n=1, m=2.

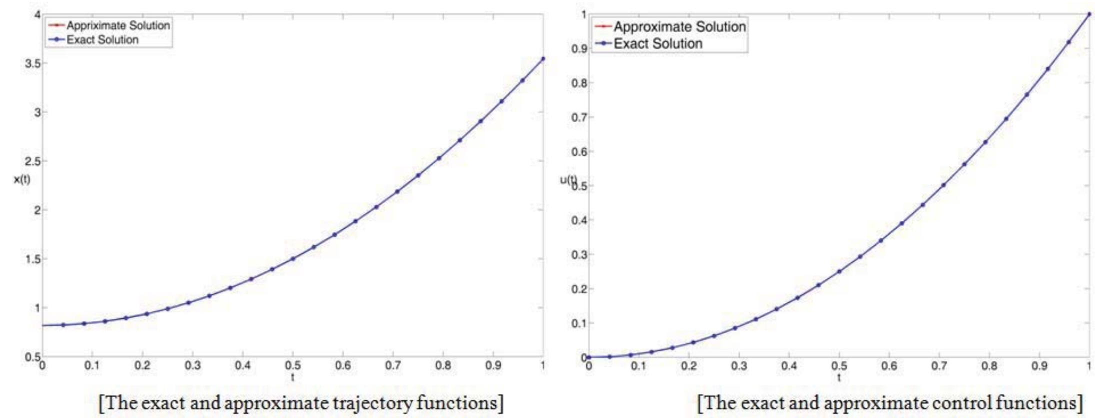


Fig. 3: Trajectory and control functions for Example 4.1, n=2, m=2.

Example 4.2:

Here in second example, we consider again the optimal control problem governed by Fredholm integral equation selected from [3]

$$Minimize\ J = \int_0^1 (x(t)-1-1.1111t)^2 + (u(t)-t^2)^2\,dt, \tag{17}$$

subject to

$$x(t) = y(t) + \int_0^1 (ts + t^2 \times u(s)) \times x(s)ds. \tag{18}$$

where $y(t) = 1$.

The Exact optimal solutions of (17)-(18) are

$$x^*(t) = 1 + 1.1111t, and \quad u^*(t) = t^2.$$

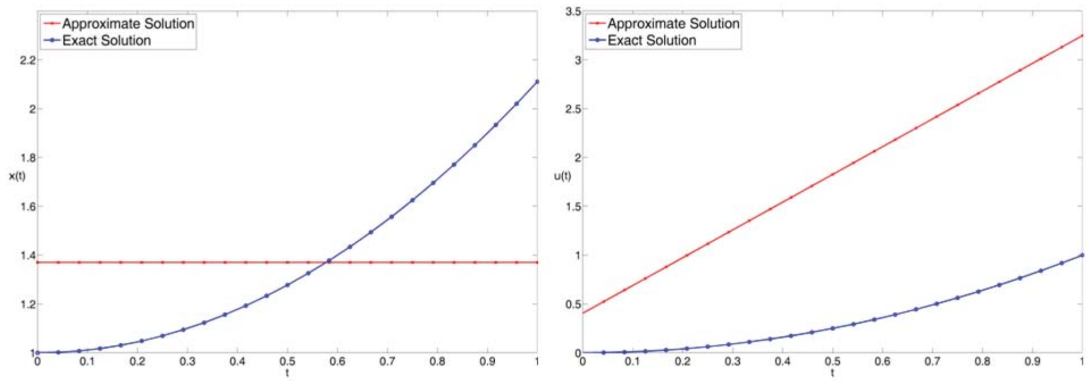
with the optimal criterion

$$J = J(x^*(t), u^*(t)) = 0.$$

Using the above method, we have the numerical results obtained in Table 2. As can be seen from Table 2 and Figures 4-6.

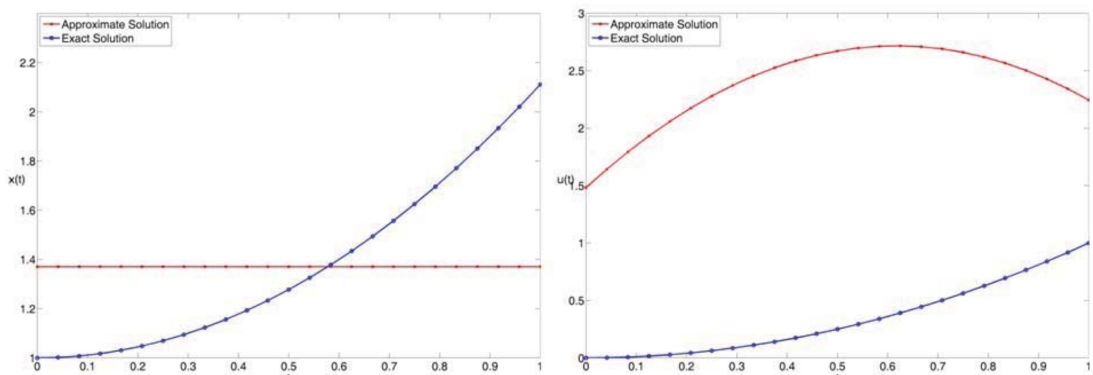
Table 2: The Approximate-Analytical results for Example 4.2

Iteration	*n*	*m*	*x(t)*	*u(t)*	*J(x(t), u(t))*
1	1	1	1.3703	$0.4054+2.8424t$	0.3097
2	1	2	1.3706	$1.4826+3.9929t-3.2309t^2$	0.5431
3	2	2	$1+10t^2/9$	t^2	0.0



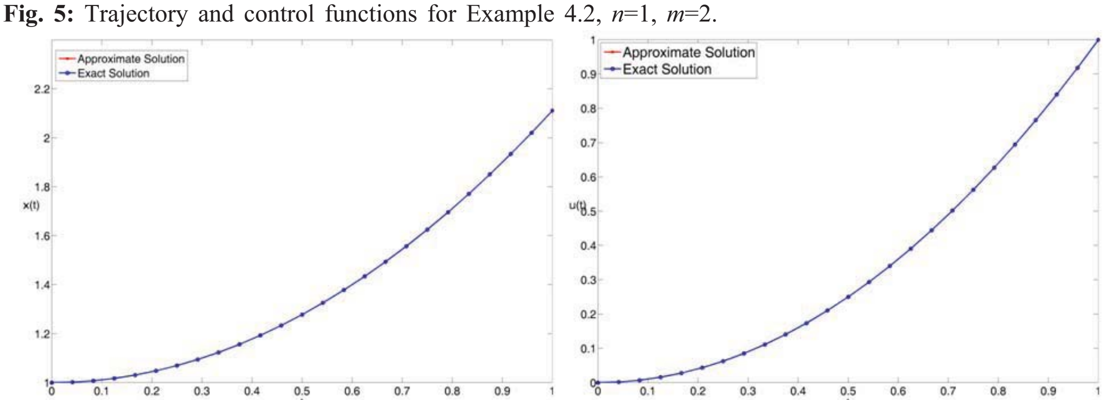
[Exact and approximate trajectory functions] [Exact and approximate control functions]

Fig. 4: Trajectory and control functions for Example 4.2,n=1, m=1.



[Exact and approximate trajectory functions] [Exact and approximate control functions]

Fig. 5: Trajectory and control functions for Example 4.2, n=1, m=2.



[Exact and approximate trajectory functions] [Exact and approximate control functions]

Fig. 6: Trajectory and control functions for Example 4.2, n=2, m=2.

6 Conclusions:

An iterative scheme for numerical solution of optimal control problems governed by Fredholm integral equations, using parametrization and Bernstein polynomials has been proposed. The convergence and uniqueness of the method has been proved. The efficiency of it, discussed in some examples.

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