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Analytical Solution for Supersonic Flow on a Conical Body of Rounded Triangle Cross Section via the Perturbation Method

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Abstract: In the present work flow over a conical body with rounded triangle cross section is studied analytically. In supersonic flow, the boundary layer is very thin and the viscous effects are negligible. The viscous terms in Navier Stokes equations are omitted and the governing equations of the problem are the Euler's Equations. The governing equations are simplified due to the geometry of the body. Spherical coordinate is considered for the problem. The Perturbation expansions are applied for flow variables, the governing equations and boundary conditions are also expanded. Applying the perturbation expansions for flow variables and substituting in the governing equations, two systems of equations are obtained. Solving the zero-order and first-order systems of equations the solution for supersonic flow on a cone of rounded triangle cross section is obtained. Results for flow variables for cone of rounded triangle cross section are presented for different semi-vertex angles of the basic circular cone and various Much numbers.

Key words: Supersonic Flow, Rounded Triangle cross section, Perturbation Method, Shock Layer

INTRODUCTION

The flow past conical bodies has been studied for many different cases. A supersonic compressible three dimensional solution is useful in design of supersonic aircrafts, missiles, rockets and etc. Taylor-Maccoll (1933) have nvestigated the steady supersonic flow past a right circular cone at zero angle of attack, they have reduced the governing equations to a single second-order nonlinear differential equation. Perturbation method is widely applied to studies of flow on conical bodie. Stone (1948,1952) applied the power series expansion for a small angle of attack and obtained a solution via perturbation method. Sims (1964) performed a numerical integration for Stone's solution. Hypersonic flows over slender pointed nose elliptic cones at zero incidence (Hemdan, H.T., 1999) is studied, the flow is sought as a small perturbation from some basic circular cone flow. In (Mascitti, R., 1972) the geometry of the cone cross sections and surface velocities are expanded in Fourier series, using the supersonic linearized conical flow theory, the flow over slender pointed cones are calculated. The analysis is similar to that of Doty and Rasmussen (1973) for obtaining solutions for flow past circular cones at small angle of attack. In this paper considering the Stone's perturbation expansions and applying them to a conical body of rounded triangle cross section we obtained an analytical solution for each of the flow parameters. The advantage of having an analytical relation for every flow parameter is the possibility of finding the quantity of a parameter at a specific point, differentiating to find an extremum point and easy integration. Through the analysis we had to find the basic cone solution which is found to be similar to the existing solutions (Stone, A.H., 1948). Therefore we presented only the perturbation solution for rounded triangle cross section. The graphs of flow parameters are presented for various cases to investigate the role of free stream velocity and slenderness of the cone in variation of each flow parameter across the shock layer. This analysis can be extended to other cross section shapes or cones with longitudinal curvature. The pressure coefficients can be obtained by quite simple integrations of pressure on the body surface and subsequently the lift and drag forces can be calculated. The results are applicable in finding lift to drag ratio which is used as a measure for choosing an appropriate cross section for a flying object.

2. Formulation and Governing Equations:

The problem is to find supersonic flow solution on a cone of rounded triangle cross section. The geometry

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of the cone and spherical coordinates adapted for this problem are presented in Fig. 1. The velocity vector in this flow is,

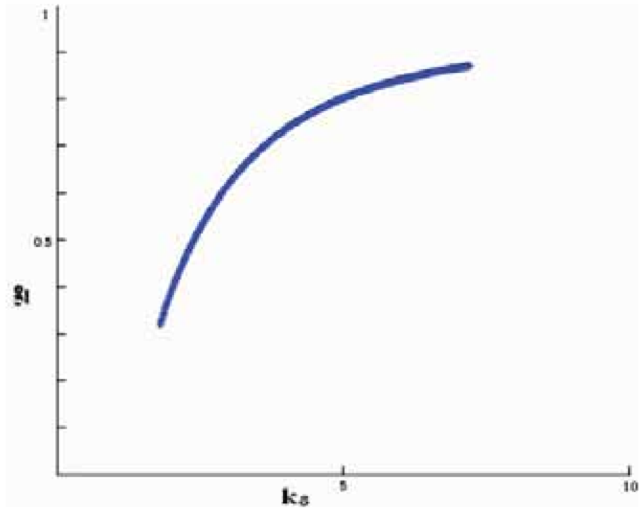

Fig. 1: Variation of shock displacement parameter versus k_δ

$$V = ue_r + ve_\theta + we_\phi \tag{1}$$

For a basic cone with circular cross section and zero angle of attack the velocity component along is zero. By changing the cross section shape to a rounded triangle the basic cone is perturbed by the following relation,

$$\theta_c = \delta - \varepsilon \cos 3\phi \tag{2}$$

In which is a parameter that determines the cross section shape and also used as the perturbation factor. For a basic cone the semi-vertex angle of shock is β and this parameter determines shock location in a spherical coordinate system, but for a rounded triangle cross section shock location shifts due to the shape of cross section, a term should be added to β to determine the new shock location,

$$\theta_s = \beta - \varepsilon g_1 \cos 3\phi \tag{3}$$

In the above relation, g1, shock displacement factor associated with the shape of cross section, is determined through the analysis.

Due to high velocities in supersonic flow, viscous effects are negligible thus the governing equations are simplified to the Euler's equations, equation of state for a perfect gas (4) and Bernoulli equation (10) are also applied.

$$2\rho u + \frac{\partial}{\partial\theta}(\rho v) + \rho v \cot\theta + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\rho w) = 0 \tag{4}$$

$$v\frac{\partial u}{\partial\theta} + w\frac{\partial u}{\partial\phi} - v^2 - w^2 = 0 \tag{5}$$

$$v\frac{\partial v}{\partial\theta} + w\frac{\partial v}{\partial\phi} - uv - w^2 \cot\theta + \frac{1}{\rho}\frac{\partial p}{\partial\theta} = 0 \tag{6}$$

$$v\frac{\partial w}{\partial\theta} + w\frac{\partial w}{\partial\phi} - uw - vw\cot\theta + \frac{1}{\rho\sin\theta}\frac{\partial p}{\partial\phi} = 0 \tag{7}$$

$$v\frac{\partial s}{\partial \theta}+\frac{w}{\sin\theta}\frac{\partial s}{\partial \phi}=0 \tag{8}$$

$$s=\text{In}\left(\frac{\rho_\infty v_\infty p}{p_\infty}\right)-\gamma\text{In}\rho=\text{In}\,(\gamma M_\infty p)-\gamma\text{In}\rho \tag{9}$$

$$\frac{1}{2}(u^2+v^2+w^2)-\frac{\gamma}{\gamma-1}\frac{p}{\rho}=\frac{1}{2}+\frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty v_\infty^2} \tag{10}$$

2-1. Perturbation Expansions and Boundary Conditions:

We assumed the first order expansions for flow over a conical body of rounded triangle cross section according to the Stone's Expansions (Stone, A.H., 1948) to be as follows. The terms with subscript 0 are the basic cone solutions which is the problem of flow over a circular cone at zero angle of attack. Terms with subscript 1 are the solutions related to perturbation of a circular cone to a rounded triangle Superimposing these solutions through the perturbation expansions, a complete solution for flow variables are obtained for the present problem.

$$u(\theta,\phi,\varepsilon)=u_0(\theta)+\varepsilon u_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{11}$$

$$v(\theta,\phi,\varepsilon)=v_0(\theta)+\varepsilon v_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{12}$$

$$w(\theta,\phi,\varepsilon)=w_0(\theta)+\varepsilon w_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{13}$$

$$p(\theta,\phi,\varepsilon)=p_0(\theta)+\varepsilon p_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{14}$$

$$\rho(\theta,\phi,\varepsilon)=\rho_0(\theta)+\varepsilon \rho_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{15}$$

$$s(\theta,\phi,\varepsilon)=s_0(\theta)+\varepsilon s_1(\theta)\cos 3\phi+O(\varepsilon^2) \tag{16}$$

Substituting the perturbation expansions in the governing equations and separating the zero-order and firstorder terms with respect to two systems of equations are obtained to be solved. The following equations constitute the zero-order system. The solution for this system is available in (Stone, A.H., 1948) and is known as the basic cone solution.

$$2\rho_0 u_o+(\rho_0 u_0)'+\rho_o v_o\cot\theta=0 \tag{17}$$

$$v_o u_0'+v_0^2=0 \tag{18}$$

$$\rho_0 v_0 v_0'-\rho_0 u_0 v_0+\frac{\partial p_o}{\partial \theta}=0 \tag{19}$$

$$v_0 s_o'=0 \tag{20}$$

$$s_o=\text{In}(p_o M_\infty^2\gamma)-\gamma\text{In}\rho_o \tag{21}$$

$$\frac{1}{2}(u_0 + v_0) + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} - \frac{1}{2} - \frac{1}{(\gamma-1)M_\infty^2} = 0 \tag{22}$$

Applying the basic cone solution the first order system of equations is reduced to terms with subscript 1. Using substitution method and elimination method, equations with respect to one of the flow parameters are obtained that can be solved analytically.

$$2(\rho_0 u_1 - \rho_1 u_0) + (\rho_1 v_0 - \rho_0 v_1)' + \cot\theta(\rho_0 v_1 - v_0 \rho_1) + \frac{3\rho_0 w_0}{\sin\theta} = 0 \tag{23}$$

$$v_0 u_1' + v_1 u_0' - 2v_0 v_1 = 0 \tag{24}$$

$$\rho_0 (v_0 v_1)' + \rho_0 v_0 v_0' + v_0(\rho_0 u_1 + \rho_1 u_0)\rho_0 u_o v_0 + p_1' = 0 \tag{25}$$

$$w_1' + \frac{u_0}{v_0} w_1 + w_1 \cot\theta - \frac{3}{\sin\theta}\frac{p_1}{\rho_0 v_0} = 0 \tag{26}$$

$$v_0 s_1' + s_0 v_1' = 0 \tag{27}$$

$$s_1 = \frac{p_1}{p_0} - \gamma\frac{\rho_1}{\rho_0} \tag{28}$$

$$\frac{1}{2}(u_0 + v_0) + (v_0 v_1 + u_0 u_1) + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} - \frac{1}{2} - \frac{1}{(\gamma-1)M_\infty^2} = 0 \tag{29}$$

To find analytical solution for flow we need boundary conditions on the body surface and at the shock. Since the shock location is not determined in this step of the analysis we express the boundary conditions at the shock as a function of g1, the shock displacement factor which itself is determined through the analysis by imposing the body boundary condition on the perturbation velocity component along θ. Finding the sock displacement factor the boundary conditions are determined and can be used in relations obtained for flow parameters. The entropy terms are found to be constants. The Rankine-Hugoniot equations, mass conservation normal to the shock and mass conservation across the shock are applied for finding the boundary conditions at the shock.

$$v_0(\delta) = 0 \tag{30}$$

$$v_1(\delta) = v_0'(\delta) = -2 \tag{31}$$

$$w_1(\beta) = -3g_1(1 - \xi_0) \tag{32}$$

$$u_1(\beta) = -\frac{\sin\beta}{2} w_1(\beta) \tag{33}$$

$$v_1(\beta) = \delta g_1 \cos\beta(-\xi_0 + 2\frac{\gamma}{\gamma - 1}) + \delta g_1 v_0'(\beta) \tag{34}$$

$$s_1 = \frac{M_1}{M_0} - \gamma\frac{\xi_1}{\xi_0} \tag{35}$$

$$p_1(\beta) = \delta g_1 p_0'(\beta) - \frac{1}{M^2\gamma}(\delta g_1 \frac{3\gamma}{\gamma - 1}\cot\beta) \tag{36}$$

Where

$$\xi_0 = \frac{(\gamma-1)K_\beta^2 + 2}{(\gamma-1)K_\beta},$$

$$\xi_0 = 2g_1 \cot\beta(\xi_0 \frac{\gamma+1}{(\gamma-1)}),$$

$$M_0 = 1 + \frac{2\gamma}{\gamma+1}(K_\beta^2 - 1),$$

$$M_1 = -\delta g_1 \frac{3\gamma K_\beta^2}{\gamma+1}\cot\beta$$

And $K_\beta = M \sin\theta$

2-2. Analytical Solution for Supersonic Flow:

The basic cone solution is presented in detail in (Stone, A.H., 1948) the final relations for flow parameters for small angles of cone and shock are presented here,

$$\frac{u_o(\theta)}{u_\infty} = 1 - \frac{\theta^2}{2} + (1-\xi_0)(\beta^2)\mathrm{In}\frac{\theta}{\beta} \tag{37}$$

$$\frac{v_o(\theta)}{u_\infty} = -\theta(1-(1-\xi_0)\frac{\beta^2}{\theta^2} \tag{38}$$

$$p_0(\theta) = \frac{1}{\gamma}(\frac{M_\infty}{\exp(s_0)})^{\frac{1}{\gamma-1}}(\frac{1}{2}(\gamma-1)(1-u_0^2-v_0^2) + \frac{1}{M_\infty^2})^{\frac{1}{\gamma-1}} \tag{39}$$

The relation between the shock and body angles is obtained by imposing the surface boundary condition(29), as follows,

$$\sigma = \frac{\beta}{\delta} = \sqrt{\frac{\gamma+1}{2} + \frac{1}{M_\infty^2\delta^2}} \tag{40}$$

From the first-order system of equations, we obtain the following relations,

$$v_1 = u_1' \tag{41}$$

$$s_1 = const \tag{42}$$

$$p_1 = p_0 s_1 + \frac{p_0 \rho_1 \gamma}{\rho_0} = s_1 p_0 + \rho_0 a_0^2 \tag{43}$$

$$\frac{\rho_1}{\rho_0} = \frac{s_1}{\gamma - 1} - \frac{1}{a_0^2}(u_0 u_1 + v_1 v_0) \tag{44}$$

$$w_1 = \frac{3 f_1}{\gamma} \frac{H_0 \theta}{\sin\theta} - \frac{3u_1}{\sin\theta} \tag{45}$$

$$F_1 = \frac{s_1}{1-\gamma} \tag{46}$$

$$H_0(\theta) = \frac{1}{I(\theta)} \int_{\beta}^{\theta} \frac{a_0^2}{v_0} I(\theta) d\theta \tag{47}$$

The integral factor is found to be,

$$I(\theta) = \exp\left(\int \frac{u_0}{v_0} d\theta\right) \tag{48}$$

$$2u_1 = v_1 \cot\theta = \frac{3w_1}{\sin\theta} + v_1' = \frac{\rho_0'}{\rho_0}\left(v_0 \frac{\rho_0}{\rho_1} - v_1\right) - v_0 \frac{\rho_1}{\rho_0} \tag{49}$$

Replacing $v1$ by (40), $w_1$by (41) and omitting the negligible terms, we can rewrite equation (48) as,

$$u_1'' = u_1' \cot\theta + \left(2 - \frac{9}{\sin\theta}\right) = -\frac{9 F_1 H_0(\theta)}{\gamma \sin^2\theta} \tag{50}$$

Solving the above differential equation for u1 and applying eqs. (41) and (45), the relations for v1 and w1 are also obtained. A particular solution for equation (50) is (51). To find the complete solution we had to find the unknown function, X1(θ) expressed by (51)

$$u_{1p} = \frac{\cos\theta}{\sin^3\theta} \tag{51}$$

$$u_1(\theta) = X_1(\theta)\left(\frac{\cos\theta}{\sin^3\theta}\right) \tag{52}$$

For small angles we have,

$$X_1(\theta) = X_1(\beta) + X_1'(\beta)\left[\frac{1}{\beta^5}\right]$$
$$\int_\beta^\theta \theta^5 d\theta +$$
$$\int_\beta^\theta \theta 5 \int_\beta^\theta -\frac{9F_1 H_0(\theta)}{\gamma}\left(\frac{1}{\theta^4}\right) d\theta d\theta \qquad (53)$$

Substituting the boundary conditions at the shock surface we obtain,

$$X_1(\beta)\left(\frac{1}{\beta^3}\right) = g_1\beta(1-\xi_0) \qquad (54)$$

Defining a dimensionless variable as $z = \theta/\delta$, relations for the velocity components and pressure are obtained as follow,

$$X_1(z) = g_1\delta^4\sigma^4\left[1-\frac{\gamma-1}{\gamma+1}-\frac{2}{(\gamma+1)M^2\delta^2\sigma^2}\right] + 4g_1\delta^3\sigma^3\left[1-\frac{\gamma-1}{\gamma+1}\right] -$$

$$\frac{4g_1\delta\sigma}{(\gamma+1)M^2}\left(\frac{\delta}{\sigma^5}\right)\left(\frac{1}{6}z^6-\frac{1}{6}\sigma^6\right) + 9\delta^3\left(\frac{1}{18}z^3-\frac{1}{18}\sigma^3\right)\frac{F_1 H_0(\theta)}{\gamma} \qquad (55)$$

$$u_1(z) = X_1(z)\left[\frac{1}{\delta^3 z^3}\right] \qquad (56)$$

$$v_1(z) = \frac{u_1'(z)}{\delta} \qquad (57)$$

$$w_1(z) = \frac{3F_1}{\gamma}\frac{H_0(\theta)}{\sin\theta} - \frac{3u_1(z)}{\sin\theta} \qquad (58)$$

Pressure can be determined by replacing u1 and v1 in equation (59)

$$p_1(\theta) = F_1 P_0(\theta) - \rho_0(\theta)(u_0 u_1 + v_0 v_1) \qquad (59)$$

The shock displacement factor is determined by satisfying the surface boundary condition (31) and then solving for g1.The shock displacement factor is plotted in Fig. 2 as a function of k_δ. For $k_\delta \rightarrow \delta$ g1 tends to zero, means that the shock tends to a circular Mach cone, $\theta = \beta$ s. When $k_\delta \rightarrow \infty$ and $\gamma \rightarrow 1$, then $\beta \rightarrow \delta$ and $g_1 \rightarrow 1$ in this case the shock embraces the body which is in agreement with hypersonic Newtonian theory.

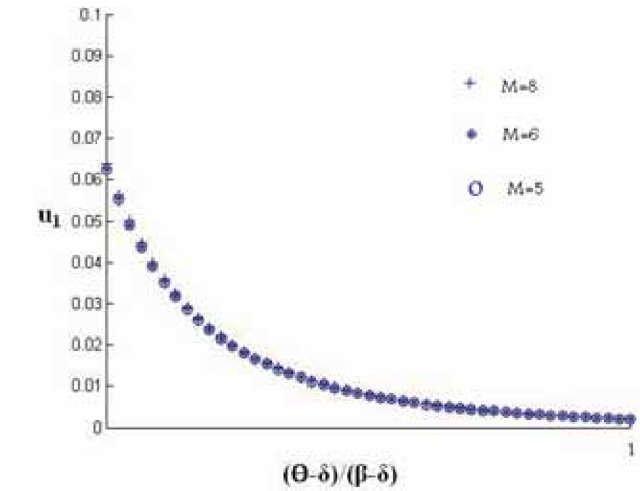

Fig. 2: Variation of u1 across shock layer (δ = 3)

$$g_1 = \frac{-2\delta}{f}$$

Where;

$$f = \frac{-4\delta^2}{(\gamma+1)M_\infty^2\sigma^4} - 3\delta^4\sigma^4\left(1-\frac{\gamma-1}{\gamma+1}-\frac{2}{(\gamma+1)M_\infty^2\delta^2\sigma^2}\right) - 12\delta^3\sigma^3\left(1-\frac{\gamma-1}{\gamma+1}\right)$$

$$+\frac{2\delta^2}{(\gamma+1)M_\infty^2\sigma^4}(1-\sigma^4)+\frac{\delta^4 H_0}{1-\gamma}\left(\frac{1}{(\gamma+1)M_\infty\beta^2-2\beta}+\frac{M_\infty^2\beta^2}{1-\gamma+2\gamma M_\infty^2\beta^2}\right)(12-\sigma^3) \quad (59)$$

Results:

Flow variables are analytically calculated, to investigate the role of Mand semi-vertex angle of the cone in variation of each of flow parameters across shock layer, $k_\delta = M\infty \sin\theta$, for small angles $k_\delta = \delta M\infty$. We considered five different values of k_δ by changing M∞ or δ . The thickness of shock layer varies by changing M∞ or δ therefore we define a dimensionless variable as $\frac{\theta-\delta}{\beta-\delta}$ to normalize the shock layer thickness. This variable varies from 0 representing the body surface to 1 for the shock surface. In Fig. 3 variation of u1 as function of $\frac{\theta-\delta}{\beta-\delta}$ and various values of k_δ is presented. Moving from the body to the shock u1 decreases. u1 variation is more sensitive to the values of M∞ rather than δ. Increasing M∞ for a constant δ the range of variation decreases but the average increases. The variation of v1 is shown in Fig. 4, the values of this variable are negative and due to the boundary condition the values on the body surface are equal to -2. The variation of v1 is more slowly for a higher k_δ . For a constant $M\infty$ increasing δ causes an increase in v1. The velocity component w1 is shown in Fig. 5. At the shock the magnitude of w1 increases as k_δ decreases and it is reverse at the body surface. w_1 variation is more intense for lower $M\infty$. The variation of p_1 is presented in Fig. 6 and is quite similar to the variation of u_1 across the shock layer. The pressure first decreases and then increases, increasing M∞, for a constant the minimum perturbation pressure gets closer to the body surface and the average pressure increases. Increasing δ for a constant $M\infty$, pressure decreases and the minimum pressure gets closer to the shock surface. When k_δ increases according to equation (40), the thickness of shock layer decreases, for this reason the range of variation of the flow parameters is smaller for higher values of k_δ.

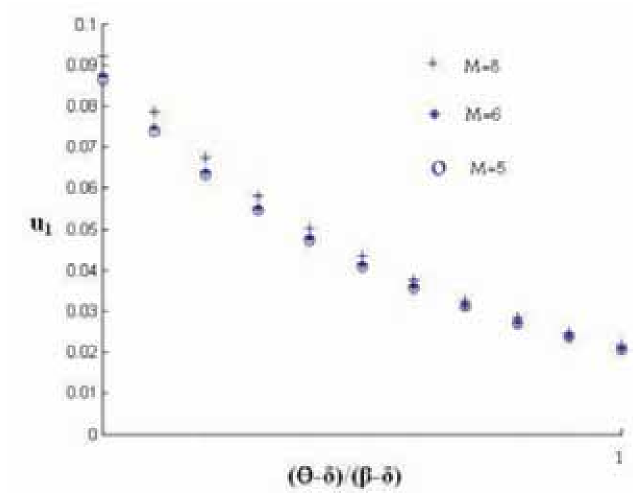

Fig. 3: Variation of u1 across shock layer (δ = 8)

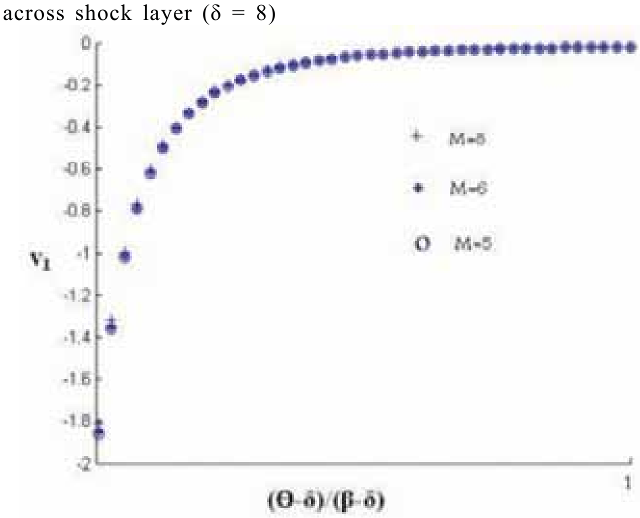

Fig. 4: Variation of v1 across shock layer (δ = 3)

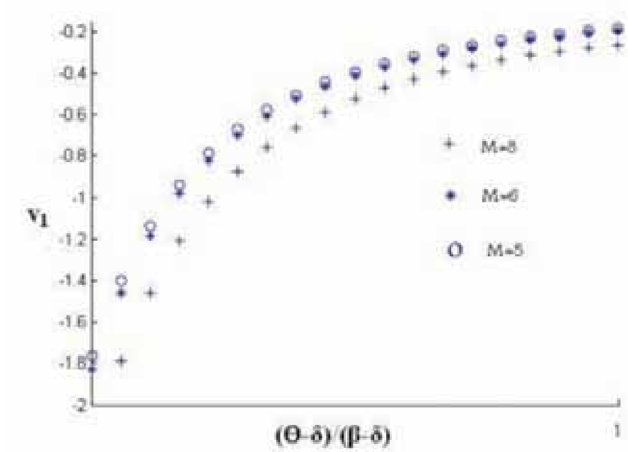

Fig. 5: Variation of v1 across shock layer (δ = 8)

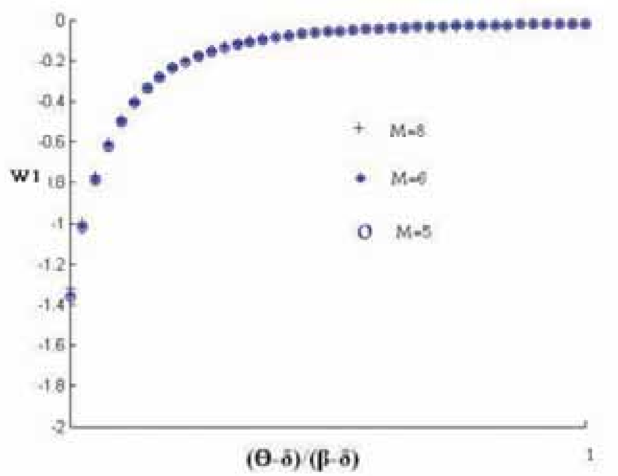

Fig. 6: Variation of w1 across shock layer (δ = 3)

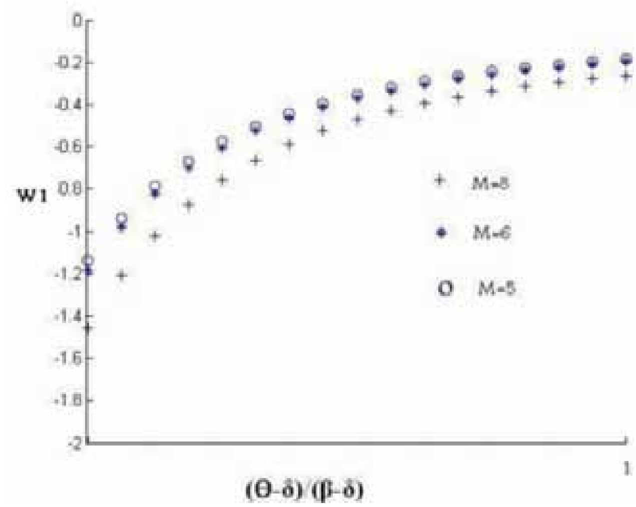

Fig. 7: Variation of w1 across shock layer δ = 8

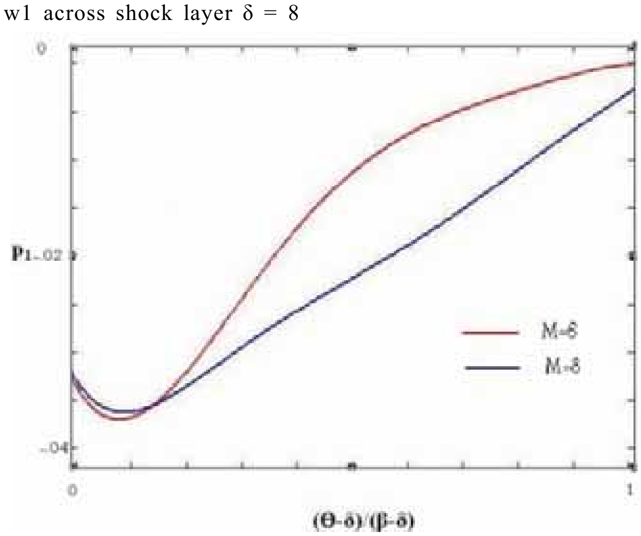

Fig. 8: Variation of p1 across shock layer (δ = 5)

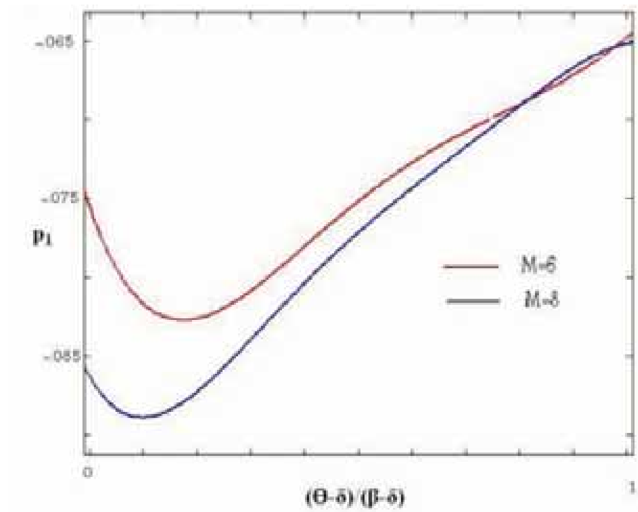

Fig. 9: Variation of p1 across shock layer (δ = 8)

***Conclusions*:**

Flow parameters over a conical body with rounded triangle cross section are obtained analytically. The governing equations are simplified to Euler's equations due to the high velocities. The perturbation method is applied to solve the problem. The graphs for the first-order perturbation velocities and pressure are presented for various values of k δ. The results show that for higher Much numbers the shock layer is thinner thus the flow parameters vary in a narrower range. This method can be extended to other cross section shapes. Integrating the pressure terms on the body normal and tangential forces can be obtained and subsequently the lift to drag ratio which is a very important parameter in design of aircrafts, missiles and other flying objects.

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