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Differential Transform Method for Solving System of Delay Differential Equation

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Abstract: In this paper, we have applied the differential transform method (DTM) to solve systems of linear or non-linear delay differential equation. A remarkable practical feature of this method its ability to solve the system of linear or non-linear delay differential equations efficiently. By using DTM, we manage to obtain the numerical, analytical, and exact solutions of both linear and non-linear equations. In comparison with the existing techniques, the DTM is a reliable method that needs less work and does not require strong assumptions.

Key words: Differential transform method, Differential inverse transform, system of delay differential equation.

INTRODUCTION

The differential transform method has been successfully used by Zhou (1986) to solve a linear and non-linear initial value problems in electric circuit analysis. In a recent year differential transform method has been used to solve one-dimensional planar Bratu problem, differential-difference equation, delay differential equations, differential-algebraic equation, integro- differential systems, (Abdel-Halim, 2007; Arikoglu, 2006; Arikoglu, 2008; Arikoglu, 2006; Ayaz, 2004; Karakoç, 2009; Kurnaz, 2005; Osmanoglu, 1986) with references therein. In this paper we reformulate DTM to solve the following system of delay differential equation (SDDE):

$$
\begin{aligned}
y_1'(t) &= f_1(t, y_1(t), \cdots, y_n(t), y_1(\zeta_j(t)), \cdots, y_n(\zeta_j(t))) \\
y_2'(t) &= f_2(t, y_1(t), \cdots, y_n(t), y_1(\zeta_j(t)), \cdots, y_n(\zeta_j(t))) \\
&\vdots \\
y_n'(t) &= f_n(t, y_1(t), \cdots, y_n(t), y_1(\zeta_j(t)), \cdots, y_n(\zeta_j(t))), \quad 0 \le t < \infty, j = 1 \cdots m \ , \\
y_i(t) &= \varphi_i(t), \ t \le 0 \ ,
\end{aligned}
\tag{1}
$$

where each equation represents the first derivative of one of the unknown functions as a mapping depending on the independent variable t , n unknown function $f_1, f_2, ..., f_n$.

Differential Transform Method:

The k^{th}-order differential transform of a function $y_i(t)$ at the point $t = t_0$ as follows (Arikoglu, 2006; Karakoç, 2009):

$$Y_i(k) = \frac{1}{k!}\left[\frac{d^k y_i(t)}{dt^k}\right]_{t=t_0} \quad , \tag{2}$$

where $y_i(t)$ is the original function, $Y_i(k)$ is the transformed function and $\frac{d^k}{dt^k}$ is the k^{th} derivative with respect to t. The differential inverse transform of $Y_i(k)$ is defined as

$$y_i(t) = \sum_{k=0}^{\infty} Y_i(k)(t - t_0)^k \tag{3}$$

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Combining equations (2) and (3) we obtain

$$y_i(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k y_i(t)}{dt^k}\right]_{t=t_0} (t-t_0)^k \tag{4}$$

The following theorems that can be deduced from equation (1) and (2) are given below, see (Arikoglu, 2006; Arikoglu, 2006; Karakoç, 2009):

Theorem 1. if $y(t) = g(t) \pm h(t)$, then $Y(k) = G(k) \pm H(k)$.

Theorem 2. if $y(t) = cg(t)$, then $Y(k) = cG(k)$.

Theorem 3. if $y(t) = \dfrac{d^k g(t)}{dt^k}$, then $Y(k) = \dfrac{(k+n)!}{k!} G(k+n)$.

Theorem 4. if $y(t) = g(t)h(t)$, then $Y(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 5. if $y(t) = t^n$, then $Y(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$.

Theorem 6. if $y(t) = g_1(t)g_2(t)\cdots g_{n-1}(t)g_n(t)$, then

$$Y(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1}).$$

Theorem 7. if $y(t) = g(t+a)$, then $Y(k) = \sum_{h_1=k}^{N} \binom{h_1}{k} a^{h_1-k} G(h_1)$, for $N \to \infty$.

Theorem 8. if $y(t) = g(\frac{t}{a}), a \geq 1$ then

$$Y(k) = \sum_{h_1=k}^{N} (-1)^{h_1-k} \frac{(a-1)^{h_1-k}}{a^{h_1}} t_0^{\,h_1-k} \binom{h_1}{k} a^{h_1-k} G(h_1), \quad \text{for } N \to \infty.$$

Theorem 9. if $y(t) = g_1(\frac{t}{a_1})g_2(\frac{t}{a_2}), \text{with } a_1, a_2 \geq 1$ then

$$Y(k) = \sum_{k_1=0}^{k} \sum_{h_1=k_1}^{N} \sum_{h_2=k-k_1}^{N} (-1)^{h_1+h_2-k} \frac{(a_1-1)^{h_1-k_1}}{a_1^{\,h_1}} \frac{(a_2-1)^{h_2-k+k_1}}{a_2^{\,h_2}}$$

$$\times t_0^{\,h_1+h_2-k} \binom{h_1}{k_1}\binom{h_2}{k-k_1} G_1(h_1)G_2(h_2), \quad \text{for } N \to \infty.$$

Numerical Examples:

In this section we present three examples, to illustrate the method for solving linear and non-linear system of delay differential equations.

Example 1. (Abdel-Naby, 2003):

Consider the system of delay differential equation

$$y_1'(t) = y_1(t) - y_2(t) + y_1(\frac{t}{2}) - e^{\frac{t}{2}} + e^{-t} \tag{5}$$

$$y_2'(t) = -y_1(t) - y_2(t) - y_2(\frac{t}{2}) + e^{-\frac{t}{2}} + e^{t} \text{ , } 0 \le t \le 1$$

with initial condition

$$y_1(0) = 1, y_2(0) = 1 \tag{6}$$

The exact solution is

$$y_1(t) = e^t, \; y_2(t) = e^{-t}$$

Using Theorems 1, 2, 3 and 8, equation (5) transforms to

$$(k+1)Y_1(k+1) = Y_1(k) - Y_2(k) + \frac{1}{2^k}Y_1(k) - F_1(k) + F_2(k)$$

$$(k+1)Y_2(k+1) = -Y_1(k) - Y_2(k) - \frac{1}{2^k}Y_2(k) + F_3(k) + F_4(k) \tag{7}$$

with initial conditions $Y_1(0)=1$ and $Y_2(0)=1$, where $F_1(k)$, $F_2(k)$, $F_3(k)$ and $F_4(k)$ are the transformed forms of the functions $f_1(t) = e^{\frac{t}{2}}$, $f_2(t) = e^{-t}$, $f_3(t) = e^{-\frac{t}{2}}$ and $f_4(t) = e^t$ respectively. It's easy to show that the differential transform of $f(t) = e^{\lambda t}$ is $F(k) = \frac{\lambda^k}{k!}$. From equation (7), we obtain

$$Y_1(1) = 1,\, Y_2(1) = -1;\, Y_1(2) = \frac{1}{2!},\, Y_2(2) = \frac{1}{2!};\, Y_1(3) = \frac{1}{3!},\, Y_2(3) = -\frac{1}{3!};\, Y_1(4) = \frac{1}{4!},\, Y_2(4) = \frac{1}{4!};$$

$$Y_1(5) = \frac{1}{5!}, Y_2(5) = -\frac{1}{5!}; Y_1(6) = \frac{1}{6!}, Y_2(6) = \frac{1}{6!}; Y_1(7) = \frac{1}{7!}, Y_2(7) = -\frac{1}{7!}; Y_1(8) = \frac{1}{8!}, Y_2(8) = \frac{1}{8!};$$

$$Y_1(9) = \frac{1}{9!}, Y_2(9) = -\frac{1}{9!}; Y_1(10) = \frac{1}{10!}, Y_2(10) = \frac{1}{10!}; Y_1(11) = \frac{1}{11!}, Y_2(12) = -\frac{1}{12!}; Y_1(13) = \frac{1}{13!},$$

$$Y_2(13) = \frac{1}{13!}; Y_1(14) = \frac{1}{14!}, Y_2(14) = -\frac{1}{14!}; Y_1(15) = \frac{1}{15!}, Y_2(15) = \frac{1}{15!}; Y_1(16) = \frac{1}{16!}, Y_2(16) = -\frac{1}{16!} \cdots.$$

Substituting these values into (4) where t_0=0, we obtain the following analytical solution

$$y_1(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!} + \frac{t^9}{9!} + \frac{t^{10}}{10!} + \frac{t^{11}}{11!} + \frac{t^{12}}{12!} + \frac{t^{13}}{13!} + \frac{t^{14}}{14!} + \frac{t^{15}}{15!} + \frac{t^{16}}{16!} + \cdots$$

$$y_2(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \frac{t^8}{8!} - \frac{t^9}{9!} + \frac{t^{10}}{10!} - \frac{t^{11}}{11!} + \frac{t^{12}}{12!} - \frac{t^{13}}{13!} + \frac{t^{14}}{14!} - \frac{t^{15}}{15!} + \frac{t^{16}}{16!} - \cdots.$$

which is formally the same as Maclaurin series of e^t and e^{-t}. In fact, the functions $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are the exact solutions of Example 1. The absolute error $e_{abs} = |y_{exact} - y_{approx}|$ of Example 1 by DTM are given in Table 1 and the absolute error of Example 1 by the spline appeoximation method (SAM) (Abdel-Naby, 2003) are given in Table 2.

Table 1 Absolute error of Example 1 at different grid points with h=0.1 by DTM

t	e_{DTM} (N=8)	e_{DTM} (N=20)
0	(0, 0)	(0, 0)
0.1	$(0,\ 1\times10^{-10})$	$(0,\ 1\times10^{-10})$
0.2	$(1\times10^{-9},\ 1\times10^{-10})$	$(1\times10^{-9},\ 1\times10^{-10})$
0.3	$(1\times10^{-9},\ 0)$	$(1\times10^{-9},\ 1\times10^{-10})$
0.4	$(1\times10^{-9},\ 8\times10^{-10})$	$(0,\ 1\times10^{-10})$
0.5	$(6\times10^{-9},\ 5.2\times10^{-9})$	$(1\times10^{-9},\ 1\times10^{-10})$
0.6	$(2.9\times10^{-8},\ 2.62\times10^{-8})$	$(1\times10^{-9},\ 0)$
0.7	$(1.19\times10^{-7},\ 1.04\times10^{-7})$	$(0,\ 1\times10^{-10})$

Table 2 Absolute error of Example 1 at different grid points with h=0.1 by SAM

t	e_{SAM} (m=4)[2]	e_{SAM} (m=5)[2]
0	(0, 0)	(0, 0)
0.1	$(7.6\times10^{-9},\ 6.4\times10^{-9})$	$(2.9\times10^{-11},\ 1.8\times10^{-10})$
0.2	$(3.8\times10^{-7},\ 5.7\times10^{-7})$	$(2\times10^{-8},\ 3\times10^{-8})$
0.3	$(7.8\times10^{-6},\ 7\times10^{-6})$	$(1.1\times10^{-6},\ 6.9\times10^{-7})$
0.4	$(5.3\times10^{-5},\ 4\times10^{-5})$	$(1.2\times10^{-5},\ 5.5\times10^{-6})$
0.5	$(2.2\times10^{-4},\ 6.4\times10^{-4})$	$(6.8\times10^{-5},\ 2.5\times10^{-5})$
0.6	$(6.5\times10^{-4},\ 4.9\times10^{-4})$	$(2.6\times10^{-4},\ 7.8\times10^{-5})$
0.7	$(1.6\times10^{-3},\ 1.3\times10^{-3})$	$(7.9\times10^{-4},\ 1.9\times10^{-4})$

Example 2: (Evans, 2004):

Consider the third-order non-linear DDE of the form:

$$y'''(t) = -1 + 2y^2(\frac{t}{2}) \quad , \quad 0 \le t \le 1 \quad , \tag{8}$$

with initial conditions

$$y(0) = 0, y'(0) = 1 \text{ and } y''(0) = 0.$$

Equation (8) can be replaced into the following system of first order non-linear DDE;

$$y_1'(t) = y_2(t)$$

$$y_2'(t) = y_3(t)$$

$$y_3'(t) = -1 + 2y_1^{\,2}(\frac{t}{2}) \quad , \quad 0 \le t \le 1 \ , \tag{9}$$

with initial conditions

$$y_1(0) = 0, y_2(0) = 1 \text{ and } y_3(0) = 0 \ . \tag{10}$$

The exact solution is,

$$y_1(t) = \sin(t), \quad y_2(t) = \cos(t) \text{ and } y_3(t) = -\sin(t) \ .$$

Using Theorems 2, 3, 4, 5 and 8, equation (9) is transformed as follows:

$$(k+1)Y_1(k+1) = Y_2(k)$$

$$(k+1)Y_2(k+1) = Y_3(k) \tag{11}$$

$$(k+1)Y_3(k+1) = -\delta(k) + 2\sum_{k_1=0}^{k} \frac{1}{2^k} Y_1(k_1)Y_1(k-k_1)$$

where $\delta(k)$ is defined in Theorem 5 with $n=0$, and the initial conditions in equation (10) trensforms to

$$Y_1(0) = 0, \ Y_2(0) = 1 \text{ and } Y_3(0) = 0 \ . \tag{12}$$

Using equations (11) and (12), we get the set of algebraic equations in $Y_1(k)$, $Y_2(k)$ and $Y_3(k)$ for k = 1,2,3,..., *N*. After solving these algebraic equations, and using the inverse transformation rule in (4), we get

$$y_1(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - O(t^{10})$$

$$y_2(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - O(t^{10})$$

$$y_3(t) = -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \frac{t^9}{9!} + O(t^{10})$$

The approximate solutions of Example 2 by DTM for $y_1(t)$, $y_2(t)$ and $y_3(t)$ are given in Tables 3, 4 and 5.

Table 3: Approximate solution of $y_1(t_i)$ for Example 2 at different grid points with *h*=0.2

t	Exact	Approximate solution by DTM with		
		N=4	*N*=5	*N*=10
0	0	0	0	0
0.2	0.1986693308	0.1986693309	0.1986693309	0.1986693309
0.4	0.3894183423	0.3894183422	0.3894183422	0.3894183422
0.6	0.5646424734	0.5646424735	0.5646424734	0.5646424734
0.8	0.7173560909	0.7173560931	0.7173560909	0.7173560909
1	0.8414709848	0.8414710096	0.8414709845	0.8414709847

Table 4: Approximate solution of $y_2(t_i)$ for Example 2 at different grid points with *h*=0.2

t	Exact	Approximate solution by DTM with		
		N=4	*N*=5	*N*=10
0	1	1	1	1
0.2	0.9800665778	0.9800665779	0.9800665779	0.9800665779
0.4	0.9210609940	0.9210609941	0.9210609941	0.9210609941
0.6	0.8253356149	0.8253356166	0.8253356149	0.8253356149
0.8	0.6967067093	0.6967067388	0.6967067092	0.6967067093
1	0.5403023059	0.5403025794	0.5403023038	0.5403023059

Table 5: Approximate solution of $y_3(t_i)$ for Example 2 at different grid points with *h*=0.2

t	Exact	Approximate solution by DTM with		
		N=4	*N*=5	*N*=10
0	0	0	0	0
0.2	-0.1986693308	-0.1986693309	-0.1986693309	-0.1986693309
0.4	-0.3894183423	-0.3894183422	-0.3894183422	-0.3894183422
0.6	-0.5646424734	-0.5646424735	-0.5646424734	-0.5646424734
0.8	-0.7173560909	-0.7173560931	-0.7173560909	-0.7173560909
1	-0.8414709848	-0.8414710096	-0.8414709845	-0.8414709847

Conclusion:

In this paper, we implemented DTM to solve a systems of delay differential equations. It is observed that DTM is an effective and reliable tool for the solution of a SDDEs. Also, we saw that, if the numerical solution of the given problems are compared with their analytical solutions, the DTM is very effective and results are quite close.

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