Australian Journal of Basic and Applied Sciences, 5(2): 222-229, 2011
ISSN 1991-8178

The Solution Some Equations by Perturbation Method

N. Taghizadeh and M. Akbari

Department of Mathematics, Faculty of Science,University of Guilan,
P.O.Box 1914, Rasht, Iran

Abstract: Perturbation method is an efficient method for obtaining solutions of Nonlinear partial differential equations. In this paper, we look for the exact solutions of some equations via perturbation method. 2000 Mathematics Subject Classification: 35J60, 35A05.

Key words: Perturbation method; Klein-Gordon equation; Fisher equation; Huxley equation; Nonlinear partial differential equation.

INTRODUCTION

The study of numerical methods for the solution of nonlinear partial differential equations (NLPDEs) has enjoyed an intense period of activity over the last 40 years from both theoretical points of view. Improvements in numerical techniques, together with the rapid advances in computer technology, have meant that many of the partial differential equations arising from engineering and scientific applications, which were previously intractable, can now, be routinely solved (Mitchell and Griffiths, 1980). A variety of powerful methods, such as the homogeneous balance method (Wang, 1995; Fan and Zhang, 1998), the hyperbolic tangent function expansion method (Parkes and Duffy, 1997; Zhang, 1999), the trial function method (Kudryshow, 1990; Xie and Tang, 2006), the sine-cosine method (C.T.Yan,1996), the Jacobi elliptic function expansion method (Liu, *et al* 2001; Z.T.fU, *et al* 2003), the superposition method (Xie,and J.S Tang, 2005), and so on, were used to investigate nonlinear dispersive and dissipative problems.

Our first interest in present work begin in implementing the perturbation method do stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The aim of this paper is to find solution of the Klein-Gordon equation, Fisher equation and Huxley equation.

1. ***The Perturbation Method:***

Let us consider given equation in the following form

$$F(u,u_t,u_x,u_{tt},u_{xx},...)=0 \tag{1}$$

If we can be written Eq. (1) as follows:

$$L^{\wedge}(D_t,D_x)u(t,x)=N[u], \tag{2}$$

where

$$L^{\wedge}(D_t,D_x)u(t,x)=\sum_{k=0}^{K}\sum_{m=0}^{M}l_{km}D_t^kD_x^m, \tag{3}$$

is linear operator of Eq. (1) and N[u] is nonlinear term of Eq .(1) , then , we look for an exact solution of Eq. (2), in the form

Corresponding Author: N. Taghizadeh Department of Mathematics, Faculty of Science, University of Guilan, P.O.Box 1914, Rasht, Iran
E-maill:taghizadeh@guilan.ac.ir

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots \tag{4}$$

Substituting (4) into Eq. (2), and collecting like powers of ε we get series of equations. Thus, we can obtain all the. $\{u_i\}_{i=1}^{n}$ ' s Setting u_i's in (4), we get the exact solution of Eq. (1).

Remark. Here ε is the grading parameter, finally we can put $\varepsilon = 1$.

2. *The Klein-gordon Equation in Special Case:*

Let us consider the Klein-Gordon equation in special case (a = b = c = 1)

$$u_{tt} - u_{xx} + u = u^3 \tag{5}$$

Thus, we can be written

$$L^{\wedge}(D_t, D_x)u(t,x) = u^3 \tag{6}$$

$$\hat{L}\left(D_1, D_X\right) = D_1^2 - D_X^2 + 1.$$

Now, we look for a solution of (6) in the form (4). Substituting (4) into (6) and collecting equal powers of ε, we obtain the following series equations:

$$\begin{aligned}
&L^{\wedge}u_1 = 0,\\
&L^{\wedge}u_2 = 0,\\
&L^{\wedge}u_3 = u_1^3,\\
&L^{\wedge}u_4 = 3u_1^2 u_2,\\
&L^{\wedge}u_5 = 3u_1^2 u_3 + 3u_1 u_2^2,\\
&L^{\wedge}u_6 = 3u_1^2 u_4 + 6u_1 u_2 u_3 + u_2^3,\\
&L^{\wedge}u_7 = 6u_1 u_2 u_4 + 3u_1 u_3^2 + 3u_2^2 u_3 + 3u_1^2 u_5,\\
&\vdots\\
&L^{\wedge}u_n = \sum_{k=2}^{n-1} u_{n-k} \sum_{l=1}^{k-1} u_l u_{k-l}, n \ge 3\\
&\vdots
\end{aligned}$$

By using $Lu_1 = 0$, gives

$$u_1 = a_1 e^{\sqrt{2}x - t} \tag{7}$$

where α_1 is real constant.

From $L^{\wedge}u_2 = 0$, we conclude ,$u_2 = 0$, thus all coefficients of u_2 will be zero. means:

$$u_{2p+2} = 0, p = 0,1,2,\dots \tag{8}$$

We have

$$\hat{L} u_3 = u_1^3. \tag{9}$$

Setting (7) into (9), we obtain

$$u_3 = \frac{a_1^3}{-8} e^{3\sqrt{2}x-3t} \tag{10}$$

$$\hat{L} u_5 = 3u_1^2 u_3 + 3u_1 u_2^2. \tag{11}$$

Applying (8)and substituting (7) and (10) into (11), gives

$$u_5 = \frac{a_1^5}{8^2} e^{5\sqrt{2}x-5t} \tag{12}$$

$$\hat{L} u_7 = 6u_1 u_2 u_4 + 3u_1 u_3^2 + 3u_2^2 u_3 + 3u_1^2 u_5. \tag{13}$$

By using (8) and substituting (7) and (10) and (12)into (13), we have solution take the form

$$u_7 = \frac{a_1^7}{-8^3} e^{7\sqrt{2}x-7t} \tag{14}$$

$$\vdots$$

Same above, we can calculate all u_{2n+1} for $n \geq 4$. Applying (4), we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \left(-\frac{1}{8}\right)^{n-1} \left(\varepsilon a_1 e^{\sqrt{2}x-t}\right)^{2n-1} = \frac{\varepsilon a_1 e^{\sqrt{2}x-t}}{1 + \frac{\varepsilon^2 a_1^2 e^{2\sqrt{2}x-2t}}{8}} \tag{15}$$

Let

$$\frac{a_1^2}{8} = 1. \tag{16}$$

Substituting (16) into (15), concludes

$$u(x,t) = \pm\sqrt{8} \sec h(\sqrt{2}x - t) \tag{17}$$

Therefore, the (17) is a exact soliton solution of the Klein-Gordon equation in special case (a =b = c = 1).

3. *The Klein-gordon Equation in the General Case:*

Let us consider the Klein-Gordon equation in the general case

$$u_{tt} - au_{xx} + bu = cu^3 \tag{18}$$

Therefore, we can be written

$$L^{\wedge}(D_t, D_x)u(t,x) = cu^3 \tag{19}$$

$$L^{\wedge}(D_t, D_x) = D_t^2 - aD_x^2 + b.$$

Now we seek a solution of (19) in the form (4). Substituting (4) into (19) and collecting equal powers of ε, we obtain the following series equations:

$$L^{\wedge}u_1 = 0,$$

$$L^{\wedge}u_2 = 0,$$

$$L^{\wedge}u_3 = cu_1^3,$$

$$L^{\wedge}u_4 = 3cu_1^2u_2,$$

$$L^{\wedge}u_5 = 3cu_1^2u_3 + 3cu_1u_2^2,$$

$$L^{\wedge}u_6 = 3cu_1^2u_4 + 6cu_1u_2u_3 + cu_2^3,$$

$$L^{\wedge}u_7 = 6cu_1u_2u_4 + 3cu_1u_3^2 + 3cu_2^2u_3 + 3cu_1^2u_5,$$

$$\vdots$$

$$L^{\wedge}u_n = c\sum_{k=2}^{n-1} u_{n-k} \sum_{l=1}^{k-1} u_l u_{k-l}, n \geq 3$$

By using $L^{\wedge}u_1 = 0,$ gives

$$u_1 = a_1 e^{x-\sqrt{a-b}t} \tag{20}$$

where α_1 is real constant.

From $L^{\wedge}u_2 = 0,$ we conclude u_2=0, thus all coefficients of u_2 will be zero. means:

$$u_{2p+2} = 0, p = 0,1,2,\ldots \tag{21}$$

We have

$$L^{\wedge}u_3 = cu_1^3. \tag{22}$$

Setting (7) into (9), we obtain

$$u_3 = \frac{a_1^3}{-8b} e^{3x-3\sqrt{a-b}t}. \tag{23}$$

$$L^{\wedge}u_5 = 3cu_1^2u_3 + 3cu_1u_2^2. \tag{24}$$

Applying (21) and setting (20) and (23) into (24), gives

$$u_5 = \frac{c^2}{8^2 b^2} a_1^5 e^{5x-5\sqrt{a-b}t}. \tag{25}$$

$$L^{\wedge} u_7 = 6cu_1u_2u_4 + 3cu_1u_3^2 + 3cu_2^2u_3 + 3cu_1^2u_5. \tag{26}$$

By using (21) and substituting (20) and (23) and (25) into (26), we have

$$u_7 = \frac{c^3}{-8^3 b^3} a_1^7 e^{7x-7\sqrt{a-b}t}. \tag{27}$$

Same above, we can calculate all u $_{2n+1}$ for n $\geq$4. Applying (4), we obtain

$$u(x,t) = \sum_{n=1}^{\infty} \left(-\frac{c}{8b}\right)^{n-1} \left(\varepsilon a_1 e^{x-\sqrt{a-b}t}\right)^{2n-1} = \frac{\varepsilon a_1 e^{x-\sqrt{a-b}t}}{1+\frac{\varepsilon^2 a_1^2 c e^{2x-2\sqrt{a-b}t}}{8b}} \tag{28}$$

Let

$$\frac{ca_1^2}{8b} = 1. \tag{29}$$

Substituting (29) into (28) , concludes

$$u(x,t) = \pm\sqrt{\frac{2b}{c}} \sec h(x-\sqrt{a-b}t) \tag{30}$$

Therefore, the (30) is a exact soliton solution of the Klein-Gordon equation.

4. *The Fisher Equation*:

Let us consider the Fisher equation

$$u_t - u_{xx} - u - u^2 = 0 \tag{31}$$

Thus, we can be written

$$L^{\wedge}(D_t, D_x)u(t,x) = -u^2 \tag{32}$$

$$L^{\wedge}(D_t, D_x) = D_t - D_x^2 - 1.$$

Now, we look for a solution of (32) in the form (4). Substituting (4) into (32) and collecting equal powers of ε, we obtain the following series equations:

$$L^{\wedge} u_1 = 0,$$

$$L^{\wedge} u_2 = -u_1^2,$$

$$L^{\wedge}u_3 = -2u_1u_2,$$
$$L^{\wedge}u_4 = -2u_1u_3 - u_2^2,$$
$$\vdots$$
$$L^{\wedge}u_n = -\sum_{k=1}^{n-1} u_k u_{n-k},$$

By using $L^{\wedge}u_1 = 0$, gives

$$u_1 = a_1 e^{\sqrt{l}x+(l+1)t} \tag{33}$$

where α_1 is real constant.
We have

$$L^{\wedge}u_2 = -u_1^2. \tag{34}$$

Setting (33) into (34), we obtain

$$u_2 = \frac{-a_1^2}{-2l+1} e^{2\sqrt{lx}+2(l+1)t} \tag{35}$$

$$L^{\wedge}u_3 = -2u_1u_2. \tag{36}$$

Setting (33) and (35) into (36), we have

$$u_3 = \frac{a_1^3}{(-3l+1)(-2l+1)} e^{3\sqrt{lx}+3(l+1)t} \tag{37}$$

$$L^{\wedge}u_4 = -2u_1u_3 - u_2^2. \tag{38}$$

By substituting (33) and (35) and (37) into (38), we have solution take the form

$$u_4 = \frac{(4l-3)a_1^4}{3(-4l+1)(-3l+1)(-2l+1)^2} e^{4\sqrt{lx}+4(l+1)t} \tag{39}$$

$$\vdots$$

Applying (4), we obtain

$$u(x,t) = \varepsilon a_1 e^{\sqrt{l}x+(l+1)t} + \varepsilon^2 \frac{-a_1^2}{-2l+1} e^{2\sqrt{lx}+2(l+1)t} + \varepsilon^3 \frac{a_1^3}{(-3l+1)(-2l+1)} e^{3\sqrt{lx}+3(l+1)t}$$

$$+\varepsilon^4 \frac{(4l-3)a_1^4}{3(-4l+1)(-3l+1)(-2l+1)^2} e^{4\sqrt{lx}+4(l+1)t} + \dots \tag{40}$$

5. *The Huxley Equation:*

Let us consider the Huxley equation

$$u_t - u_{xx} - (a+1)u^2 + u^3 + au = 0, \tag{41}$$

Therefore, we can be written

$$L^{\wedge}(D_t, D_x)u(t,x) = -u^3 + (a+1)u^2, \tag{42}$$

$$L^{\wedge}(D_t, D_x) = D_t - D_x^2 + a.$$

Now we seek a solution of (42) in the form (4). Substituting (4) into (42) and collecting equal powers of ε, we obtain the following series equations:

$$L^{\wedge}u_1 = 0,$$

$$L^{\wedge}u_2 = (a+1)u_1^2,$$

$$L^{\wedge}u_3 = -u_1^3 + 2(a+1)u_1u_2.$$

$$u_3 = \frac{2a^2+5a+4}{(-4a-6)(-2-a)}a_1^3e^{3x+3(1-a)t}.$$

$$L^{\wedge}u_4 = -3u_1^2u_2 + (a+1)(u_2^2 + 2u_1u_3).$$

$$u_4 = \frac{-4a^4-52a^3-134a^2-144a-58}{(-5a-12)(-2-a)^2(-4a-6)}a_1^4e^{4x+4(1-a)t}.$$

$$u(x,t) = \varepsilon a_1e^{x+(1-a)t} + \varepsilon^2\frac{a+1}{-2-a}a_1^2e^{2x+2(1-a)t} + \varepsilon^3\frac{2a^2+5a+4}{(-4a-6)(-2-a)}a_1^3e^{3x+3(1-a)t}$$

$$+\varepsilon^4\frac{-4a^4-52a^3-134a^2-144a-58}{(-5a-12)(-2-a)^2(-4a-6)}a_1^4e^{4x+4(1-a)t} + \cdots$$

$$L^{\wedge}u_3 = -u_1^3 + 2(a+1)u_1u_2,$$

$$L^{\wedge}u_4 = -3u_1^2u_2 + (a+1)(u_2^2 + 2u_1u_3),$$

$$\vdots$$

$$L^{\wedge}u_n = -\sum_{k=2}^{n-1}\sum_{l=1}^{n-1}u_{n-k}u_lu_{k-l} + (a+1)\sum_{k=1}^{n-1}u_ku_{n-k}, n \geq 2$$

By using $L^{\wedge}u_1 = 0$, gives

$$u_1 = a_1e^{x+(1-a)t} \tag{43}$$

Where α_1 and a are real constants

We have

$$\hat{L} u_2 = (a+1)u_1^2. \tag{44}$$

Setting (43) into (44), we obtain

$$u_2 = \frac{a+1}{-2-a} a_1^2 e^{2x+2(1-a)t}. \tag{45}$$

$$\hat{L} u_3 = -u_1^3 + 2(a+1)u_1 u_2. \tag{46}$$

Setting (43) and (45) into (46), gives

$$u_3 = \frac{2a^2+5a+4}{(-4a-6)(-2-a)} a_1^3 e^{3x+3(1-a)t}. \tag{47}$$

$$\hat{L} u_4 = -3u_1^2 u_2 + (a+1)(u_2^2 + 2u_1 u_3). \tag{48}$$

By substituting (43) and (45) and (47) into (48) we have solution take the form

$$u_4 = \frac{-4a^4 - 52a^3 - 134a^2 - 144a - 58}{(-5a-12)(-2-a)^2(-4a-6)} a_1^4 e^{4x+4(1-a)t}. \tag{49}$$

Applying (4), we obtain

$$u(x,t) = \varepsilon a_1 e^{x+(1-a)t} + \varepsilon^2 \frac{a+1}{-2-a} a_1^2 e^{2x+2(1-a)t} + \varepsilon^3 \frac{2a^2+5a+4}{(-4a-6)(-2-a)} a_1^3 e^{3x+3(1-a)t}$$
$$+\varepsilon^4 \frac{-4a^4 - 52a^3 - 134a^2 - 144a - 58}{(-5a-12)(-2-a)^2(-4a-6)} a_1^4 e^{4x+4(1-a)t} + \cdots \tag{50}$$

REFERENCES

Mitchell, A.R., D.F. Griffiths, 1980. The Finite Difference Method in Partial Equations.John Wiley Sons.

Wang, M.L., 1995. Solitary wave solution for variant Boussinesq equations. Phys. Lett A., 169-172.

Fan, E.G., H.Q. Zhang, 1998. The homogeneous balance method for solving nonlinear soliton equations.Acta Phys.Sin; (in Chinese) 47(3): 353-362.

Parkes, E.J., B.R. Duffy, 1997. Travelling solitary wave solutions to a compound Kdv-Burgers equation.Phy.Lett.A., 229: 217-220.

Zhang, J.F., 1999. New exact solitary wave solutions of the KS equation. Int.J.Theor. Phys., 38(6): 1829-1834.

Kudryshow, N.A., 1990. Exact solutions of the generalized kuramoto Sivashinsky equations.Phys. Lett.A., 147: 287-292.

Xie, Y.X., J.S. Tang, 2006. A unified trial function method in finding the explicit and exact solutions to three NPDEs.Phys.Scr., 74(8): 197-200.

Yan, C.T., 1996. A simple transformation for nonlinear waves. Phys. Lett. A., 227: 77-84.

Liu, S.K., Z.T. Fu and S.D. Liu, 2001. Expansion method about the Jacobi elliptic function and its applications to nonlinear wave equations. Acta. Sin. (in Chinese) 50(11): 2068-2073.

fU, Z.T., S.D. Liu, K.S. Liu, 2003. Solutions to Generalized mKdV equation.Commun.Theor.Phys., 40(6): 641-644.

Xie, Y.X., J.S. Tang, 2005. New solitary wave solutions to the KdV-Burgers equation. Int. J.The-or.Phys., 44(3): 293-301.