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A New Reliable Algorithm Based on Homotopy-perturbation Method for Power-law Fin-type Problems

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Abstrac: In this paper, we applied the homotopy-perturbation method (HPM) to evaluate the temperature distribution of a straight rectangular fin with power-law temperature dependent surface heat flux. The time interval is divided into several subintervals and the HPM solutions are applied successively over these reduced time intervals. Comparisons between the 13-term HPM solution on a fixed time step and 6-term HPM solution on three different time steps are made. The results indicates that the obtained solution using different time step is quite accurate when only the six terms are used in the series expansion.

Key words: Homotopy-perturbation method, Nonlinear differential equations, Fin, Extended surface.

INTRODUCTION

Fins are extensively used to enhance the heat transfer between a solid surface and its convective, radiative, or convective radiative surface (Kern, 1972). Finned surfaces are widely used, for instance, for cooling electric transformers, the cylinders of air-craft engines, and other heat transfer equipment. In many applications various heat transfer modes, such as convection, nucleate boiling, transition boiling, and film boiling, the heat transfer coefficient is no longer uniform. A fin with an insulated end has been studied by many investigators (Lai,, 1967; Sen, S. Trinh, 1986; Unal, 1987). Most of them are immersed in the investigation of single boiling mode on an extended surface. Under this circumstances very recently, Chang (2005) applied standard Adomian decomposition method for all possible types of heat transfer modes to investigate a straight fin governed by a power-law-type temperature dependent heat transfer coefficient using 13 terms. Liu (1995) found that Adomian method could not always satisfy all its boundary conditions leading to an error at its boundaries.

The objective of this paper is to present an alternative procedure based on homotopy-perturbation method (HPM), to determine the temperature distribution of a straight rectangular fin with temperature dependent surface heat flux for all possible types of heat transfer considering fewer terms in the series expansion. The HPM was first proposed by He in (1998) and was further developed and improved by He (1999). In recent years, much attention has been given to the study of the homotopy-perturbation method (HPM) (Chowdhury, 2008; Ganji, 2006; He, 2000; J.H. He, 2006) for solving a wide range of problems whose mathematical models yield differential equation or system of differential equations.

HPM deforms a difficult problem into a set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers, cf. (Chowdhury, 2007; Noor, 2007). Very recently, Chowdhury *et al.* (2008) were the first to successfully apply the multistage homotopy-perturbation method (MHPM) to the chaotic Lorenz system. In this study the cases with negative exponents are especially noted and results are compared with (chang, 2005).

II. Model Problem:

The temperature distribution of a straight rectangular fin with a power-law temperature dependent surface heat flux can be determined by the solutions of a one-dimensional steady state heat conduction equation, which in dimensionless form, is given by, (chang, 2005),

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$$\frac{d^2\theta}{dx^2} - N^2\theta^{n+1} = 0. \tag{1}$$

where the axial distance x is measured from the fin tip, µ(x) is the temperature, and N is the convective -conductive parameter of the fin. The values of n vary in a wide range between ¡4 and 5 depending on the mode of boiling (cf.,(Liaw, 1994). For example, the exponent n may take the respective values ¡4;¡0:25; 0; 2, and 3, depending on whether the fin subject to transition boiling, laminar film boiling or condensation, convection, nucleate boiling, and radiation into free space at zero absolute temperature. For simplicity, consider the fin tip is insulated and the boundary conditions to Eq. (1) can be written as

$$\frac{d\theta}{dx}(0) = 0.\theta(1) = 1 \tag{2}$$

III. Homotopy-perturbation Method:

To illustrate the basic ideas of this method, we consider the following general nonlinear differential equation, (J.H. He, 2006)

$$A(u) - f(r) = 0, r \in \Omega \tag{3}$$

with boundary conditions

$$B(u, \partial u / \partial n) = 0\, r \in \Gamma \tag{4}$$

where A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, and ¡ is the boundary of the domain -.

The operator A can be generally divided into linear and nonlinear parts, say L and N. Therefore (3) can be written as

$$L(u) + N(u) - f(r) = 0. \tag{5}$$

We construct a homotopy $v(r, p) : \Omega \times [0,1] \rightarrow \Re$ which satisfies

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{6}$$

where p is called homotopy parameter.

According to the HPM, the approximation solution of (6) can be expressed as a series of the powers of p, i.e.

$$u = \lim_{P \to 1} v = v_0 + v_1 + v_2 + ... \tag{7}$$

IV. Application of HPM:

The approximate analytical solution to (1)–(2) was presented by Chang (2005) using the analytic Adomian decomposition method (ADM). Sometimes it is a very intricate problem to calculate the so-called Adomian polynomials involved in ADM. Another analytic method which has been shown to be much simpler than the ADM is called the homotopy-perturbation method (HPM). In this section, we shall demonstrate an alternative approach of the HPM to handle Eqs. (1)–(2). To do so, we first construct a homotopy $v(r,p) : \Omega \times [0,1] \rightarrow \Re$ which satisfies

$$\frac{d^2\theta}{dx^2} - \frac{d^2 y0}{dx^2} + p\left(\frac{d^2 y0}{dx^2} - N^2\,\theta^{n+1}\right) = 0. \tag{8}$$

Suppose the solution of Eq. (1) has the form:

$$\theta(x)=u_0(x)+pu_1(x)+p^2u^2(x)+...,\tag{9}$$

and let us choose the initial approximation as

$$u_0(x)=y_0(x)=\theta(0)=c,\tag{10}$$

where c is to be determined.
Substituting (9) into (8) and equating the terms with identical powers of p, we get the following system of linear differential equations:

$$\frac{d^2u_1}{dx^2}+\frac{d^2y_0}{dx^2}-N^2\,\theta^{n+1}\quad u_1(0)=0,\frac{du_1}{dx}(0)=0,$$

$$\frac{d^2u_2}{dx^2}-N^2(n+1)u_1\,u_0^n=0,\qquad u_2(0)=0,\frac{du_2}{dx}(0)=0,$$

$$\frac{d^2u_3}{dx^2}-N^2(n+1)\;u_2\,u_0^n-\frac{1}{2}N^2\,n(n+1)u_1^2\,u_0^{n-1}=0,\qquad u_3(0)=0,\frac{du_3}{dx}(0)=0,$$

$$\frac{d^2u_4}{dx^2}-N^2(n+1)\left[u_0^n\,u_3+n\,u_0^{n-1}\,u_1u_2+\frac{1}{6}n(n-1)u_0^{n-2}\,u_1^3\right]=0,$$

$$u_4(0)=0,\frac{du_4}{dx}(0)=0,$$

etc
Solving the above equations, we have

$$u_1(x)=\frac{1}{2}c^{n+1}\,N^2\,x^2,$$

$$u_2(x)=\frac{1}{24}c^{2n+1}(n+1)N^4\,x^4,$$

$$u_3(x)=\frac{1}{720}(n+1)(4n+1)c^{3n+1}\,N^6\,x^6$$

$$u_4(x)=\frac{1}{40320}(n+1)(34n^2+5n+1)c^{4n+1}\,N^8\,x^8$$

$$u_5(x)=\frac{1}{3628800_{c^{5n+1}N^{10}x^{10},}}(n+1)(496n^3-66n^2+69n+1)$$

etc.
According to (9) and the assumption p = 1, the six-term approximate solution to (1) is

$$\theta \approx c+\frac{1}{2}c^{n+1}\,N^2\,x^2+\frac{1}{24}c^{2n+1}(n+1)N^4\,x^4+\frac{1}{720}(n+1)(4n+1)$$

$$c^{3n+1}\,N^6\,x^6+\frac{1}{40320}\,(n+1)\times(34n^2+5n+1)c^{4n+1}\,N^8\,x^8+\frac{1}{3628800}(n+1)$$

$$\times(496n^3 - 66n^2 + 69n + 1)\, c^{5n+1}\, N^{10}\, x^{10} \tag{11}$$

The complete solution is obtained once the constant c is determined by imposing the second boundary condition given by Eq. (2). Note that the value of c must lie in the interval (0; 1) to represent the temperature at the fin tip (Chang, 2005).

Now, we apply the HPM as an algorithm for approximation the dynamics response in a sequence of time intervals (time step) [0; t1); [t1; t2); [t2; t3):::[tn¡1; tf) such that the initial condition in [tp; tp+1) is taken to be the condition at tp. This strategy was first hinted in (Adomian, 1988). For practical computations, a finite number of terms in the series

$$\phi_i(x) = \sum_{k=0}^{i-1} \theta_k \tag{12}$$

are used in a time step procedure just outlined.

RESULTS AND DISCUSSIONS

Now we consider the nonlinear Eq. (1). Taking the actual physiological data in (Chang, 2005) and incorporating the recursive algorithm (10)-(11), we find out the 6-term and 13-term HPM solutions for various values of $-4 \leq n, \leq 5$. For N = 1, n = 5, 13-term and 6-term approximate HPM solutions are respectively,

$$\varphi^{13} = 0{:}81620 + 0{:}14780x^2 + 0{:}02675x^4 + 0{:}00678x^6 + 0{:}00183x^8 + 0{:}00051x^{10} + 0{:}00014x^{12} + 0{:}00004x^{14} + 0{:}00001x^{16} + 0{:}000004x^{18} + 0{:}000001x^{20} + 0{:}0000003x^{22} + 0{:}0000001x^{24}; \tag{13}$$

and

$$\varphi_6 = 0{:}81623 + 0{:}14785x^2 + 0{:}02678x^4 + 0{:}00679x^6 + 0{:}00183x^8 + 0{:}00051x^{10} \tag{14}$$

similar expression have been obtained for other values of $-4 \leq n \leq 5$.

The HPM algorithm is coded in computer algebra package Maple and the Maple environment variable Digits is set to 16 in all calculation done for the current problem. Obviously the accuracy of our present 6-term HPM solution is verified by the 13-term HPM solution. Fig. 1 shows the temperature profiles for several assigned values of n at N = 1 given by Eq. (14) on the time step h = 0:01. All these numerical results are in very good agreement with the 13-term HPM solutions. As indicated in Eq. (14), the temperature along the fin is expressed in an explicit function of position x. Thus the temperature profile can be easily obtained for any exponent value n. The characteristics of temperature profiles have been discussed by Dul'kin and Garas'ko (2002) and Liaw and Yeh (Liaw, 1994). The former used the hyper-geometric formulas to determine the profiles and the latter derived an inversed form for the temperature distribution along the fin, and then evaluated the profile via an iterative procedure. Min-Hsing Chang (2005) used ADM to analyze the thermal characteristics of straight rectangular fin using 13 terms in the series expansion. The present results are consistent with both of them while with more straightforward process and less computation and only 6 terms used in series expansion. To make a proper comparison, we first determine the accuracy of HPM for the solution of Eq. (1) for different time steps. In Table I, we compare HPM solutions between 13-term for fixed time step, and 6 terms for three different time steps h = 0:1, h = 0:01 and h = 0:001 and when n = -0:5. In Table II we present the absoluteerrors between 6 terms HPM solutions at three different time steps h = 0:1, h = 0:01 and h = 0:001 , and 13 terms HPM solutions for fixed time step when n = ¡0:5. In Table III, we compare HPM solutions between 13-term for fixed time step, and 6 terms for three different time steps h = 0:1, h = 0:01 and h = 0:001 when n = 5. In Table IV we present the absolute errors between 6 terms HPM solutions at three different time steps h = 0:1, h = 0:01 and h = 0:001 , and 13 terms HPM solutions for fixed time step when n = 5. On the time steps h = 0:1, h = 0:01 and h = 0:001, the 6-term HPM solutions match with 13-term HPM solutions at least 4 decimal places. This suggests that the present HPM solutions using only 6 terms on the time step h = 0:01 are accurate enough when n = -0:5 and n = 5 . Similar conclusions have been obtained for the other values of -4 · n · 5. Several cases with variations of parameters N and n are also tested and it can be conclude that the use of 6 terms in Eq. (14) is sufficient to yield accurate results. Obviously, the present method gives fast and accurate results instead of complicated numerical integration and iteration procedure.

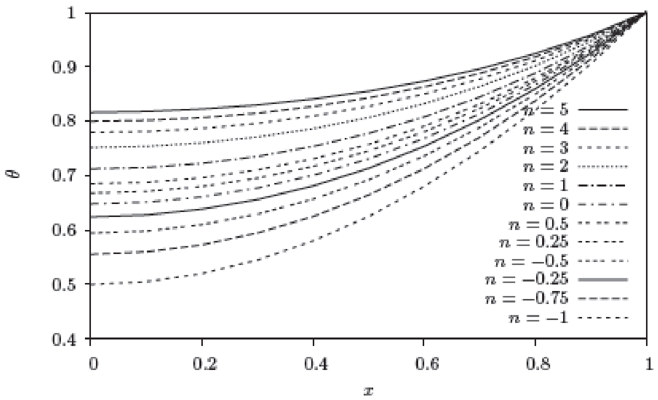

Fig. 1: The numerical solutions Á6 at time step, h = 0:01 and N = 1.

Table I: Comparison Between 13-term Hpm Solution and 6-term Hpm Solutions at Different Time Steps When N = 05.

x	φ_{13}	φ_6		
		h = 0.1	h = 0.01	h = 0.001
0.0	0.5944461335	0.5944515788	0.5944515788	0.5944515788
0.1	0.5983032326	0.5983086955	0.5983086955	0.5983086955
0.2	0.60989947616	0.6099049921	0.6099049921	0.6099049921
0.3	0.6293093861	0.6293149903	0.6293149903	0.6293149903
0.4	0.6566561362	0.6566618639	0.6566618639	0.6566618639
0.5	0.6921100992	0.6921159838	0.6921 159838	0.6921159838
0.6	0.7358869771	0.7358930398	0.7358930398	0.7358930398
0.7	0.7882456461	0.7882518437	0.7882518437	0.7882518437
0.8	0.8494858433	0.8494918787	0.8494918787	0.8494918787
0.9	0.9199458118	0.9199505587	0.9199505587	0.9199505587
1.0	0.9999999999	0.9999999999	0.9999999999	0.9999999999

Table II: Absolute Errors Between 13-term Hpm, and 6-term Hpm at Different Time Steps When N = -0:5.

x	φ_{13} - φ_6		
	h = 0:1	h = 0:01	h = 0:001
0	5.445E-06	5.445E-06	5.445E-06
0.1	5.463E-06	5.463E-06	5.463E-06
0.2	5.516E-06	5.516E-06	5.516E-06
0.3	5.604E-06	5.604E-06	5.604E-06
0.4	5.728E-06	5.728E-06	5.728E-06
0.5	5.885E-06	5.885E-06	5.885E-06
0.6	6.063E-06	6.063E-0	6 6.063E-06
0.7	6.198E-06	6.198E-06	6.198E-06
0.8	6.035E-06	6.035E-06	6.035E-06
0.9	4.747E-06	4.747E-06	4.747E-06
1	3.000E-16	1.000E-15	1.700E-14

Table III : Comparison Between 13-term Hpm Solutions and 6-term Hpm Solutions at Different Time Steps When N = 5.

x	φ_{13}	φ_6		
		h = 0.1	h = 0.01	h = 0.001
0.0	0.8161488485	0.8162264350	0.8162264350	0.8162264350
0.1	0.8176292306	0.8177076630	0.8177076630	0.8177076630
0.2	0.8221028946	0.8221838988	0.8221838988	0.8221838988
0.3	0.8296699290	0.8297553375	0.8297553375	0.8297553375
0.4	0.8405060050	0.8405978392	0.8405978392	0.8405978392
0.5	0.8548771743	0.8549777252	0.8549777252	0.8549777252
0.6	0.8731640540	0.8732758154	0.8732758154	0.8732758154
0.7	0.8959000779	0.8960248088	0.8960248088	0.8960248088
0.8	0.9238322087	0.9239663936	0.9239663936	0.9239663936
0.9	0.9580195857	0.9581375940	0.9581375940	0.9581375940
1.0	1.000000000	0.9999999999	1.000000000	1.000000000

Table IV: Absolute Errors Between 13-term Hpm, and 6-term Hpm at Different Time Steps When N = 5.

x	φ_{13} - φ_6		
	h = 0:1	h = 0:01	h = 0:001
0	7.759E-05	7.759E-05	7.759E-05
0.1	7.843E-05	7.843E-05	7.843E-05
0.2	8.100E-05	8.100E-05	8.100E-05
0.3	8.541E-05	8.541E-05	8.541E-05
0.4	9.183E-05	9.183E-05	9.183E-05
0.5	1.006E-04	1.006E-04	1.006E-04
0.6	1.118E-04	1.118E-04	1.118E-04
0.7	1.247E-04	1.247E-04	1.247E-04
0.8	1.342E-04	1.342E-04	1.342E-04
0.9	1.180E-04	1.180E-04	1.180E-04
1	5.000E-16	1.000E-15	1.000E-14

Conclusion:

In this task, The modified homotopy-perturbation method has been applied to evaluate the temperature distribution of a straight rectangular fin with temperature dependent surface heat flux for all possible types of heat transfer.It was shown that the method is reliable, efficient and requires less computations. It would be useful to apply this method to a variety of nonlinear heat conduction problems, and helpful for engineer to analyze highly nonlinear systems considering fewer terms. Therefore it is an alternative approach for practicing engineers to use instead of the more sophisticated analytical methods outlined in (Dul'kin, 2002).

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