Australian Journal of Basic and Applied Sciences, 5(4): 60-66, 2011
ISSN 1991-8178

Solution for Partial Differential Equations Involving Logarithmic Nonlinearities

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Abstract: In this paper, a modification of He's variational iteration method by using r terms of Taylor's series is applied for finding the solution of Kolmogorov-Petrovskii-Piskunov and Klein-Gordon equations with logarithmic nonlinearities. This modification cause to the new application of the variational iteration method for equations with logarithmic nonlinear part. To show the efficiency of the method, several examples are presented.

Key words: Component; variational iteration method; nonlinear PDE's; Taylor's series; Kolmogorov-Petrovskii-Piskunov equation; Klein-Gordon equation; correction functional; Lagrange multiplier.

INTRODUCTION

Mathematical modeling of many physical systems leads to nonlinear partial differential equations in various fields of physics and engineering. That's why recently nonlinear partial differential equation has attracted scientists' attention. Several methods were proposed to solve partial differential equations which are mostly applicable just for linear or polynomial's nonlinearity type equations. In this work we considered the variational iteration method (VIM), which is a useful instrument for solving partial differential equations with polynomials nonlinearity. This method first proposed by He (2000, 1997, 1999, 1998) and was applied for solving many kinds of situations. We have made a modification on the (VIM) in order to apply it for those PDE's with logarithmic nonlinearities. In fact the logarithmic part is replaced with an appropriate polynomial. To show the efficiency, we employed this modified method for solving the Klein-Gordon equation (Yusufoulu, E., 2008) and the Kolmogorov-Petrovskii-Piskunov (KPP) equation (Rottschäfer, V. and C.E. Wayne, 2001; Kolmogorov, A.N., 1973), with logarithmic nonlinear parts, which are two applicable equations. The convergence of variational iteration method was discussed in (Zaid, M., 2010).

Klein-Gordon was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons.

One of the best known model equations with dissipation is the equation suggested in 1937 by Kolmogorov, Petrovskii and Piskunov. This equation describes such phenomena as combustion (physics) and propagation of concentration waves. This method could be used for any nonlinear partial differential equations involving logarithmic nonlinearities. The results reveal that the method is very effective and convenient.

In section 2 variation iteration method is described, section 3 is devoted to application of this method for solving Kolmogorov-Petrovskii-Piskunov and Klein-Gordon equations with logarithmic nonlinearity and in section 4 numerical examples are presented.

Variational Iteration Method:

To illustration the basic concepts of the variational iteration method (VIM) we consider the following general nonlinear system:

$$Lu(x,t) + Nu(x,t) = g(x,t) \tag{1}$$

Where L is the linear operator and N is the nonlinear operator, and $g(x, t)$ is the inhomogeneous term. In the variational iteration method, where a correction function for (1) can be written:

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$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\big(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\big)d\tau \tag{2}$$

It is obvious that the successive approximation u_n, n $\geq$ 0 can be established by determining λ, a general Lagrange multiplier, which can be identified optimally via the variational theory(Inokuti, M., 1978). The function $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified via integration by parts. The successive approximations $u_{n+1}(x,t)$ of the solution *u(x,t)* will be readily obtained upon using the Lagrange multiplier obtained by using any selective function $u_0(x,t)$.

Main Part:

In this section we introduce a new method for solving nonlinear PDE's. We will consider, amongst others, the nonlinear Klein-Gordon equation (Dmitrievich, A., 2004), of the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + F(u), \quad 0 < x < \alpha, \qquad t > 0. \tag{3}$$

where *F* is a nonlinear part of equation that could be logarithmic type.

Theorem 3.1:

Taylor's theorem with Lagrange remainder (Kincade, D., W. Cheney, 1996):

Suppose that $f \in C^n[a,b]$ and if $f^{(n+1)}$ exists on *(a,b)* then for any points *c* and *x* in [*a, b*],

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(c)(x-c)^k + E_n(x),$$

Where, for some points ξ between *c* and *x*

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}.$$

In this method, we use Taylor's polynomial with remainder instead of a nonlinear part *F(u)*, as follows:

$$F(u) = P_r(u) + E_r(u),$$

such that

$$P_r(u) = \sum_{i=0}^{r} m_i u^i,$$

where

$$m_i = \frac{F^{(i)}(a)}{i!}.$$

As a result, we obtain the following formula

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + P_r(u) + E_r(u)$$

by omitting the remainder we consider a new Klein-Gordon equation with polynomial nonlinearities:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + P_r(u).$$

Therefore by means of (VIM) we get to the following correctional function

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\Big(\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) - P_r\big(\tilde{u}_n(x,\tau)\big)d\tau, \quad (4)$$

Making the above correction functional stationary, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\tau)\Big(\frac{\partial^2}{\partial \tau^2} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) - P_r\big(\tilde{u}_n(x,\tau)\big)d\tau,$$

Which yields the following stationary conditions:

$$\lambda''(t) - \lambda(t) = 0,$$

$$1 - \lambda'(t)|_{\tau=t} = 0,$$

$$\lambda(\tau)|_{\tau=t} = 0.$$

The general Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = \sinh(\tau - t).$$

By noting that $\delta\tilde{u}_n = 0$ and using $u_0(x, t)$ as a selective function, $u(x, t)$ will be obtained.

The Kolmogorov-Petrovskii-Piskunov equation which is

$$u_t - u_{xx} - F(u) = 0$$

where F is a nonlinear part of equation that could be logarithmic type.
We use Taylor's polynomial with remainder instead of a nonlinear part $F(u)$, as follows:

$$F(u) = P_r(u) + E_r(u),$$

such that

$$P_r(u) = \sum_{i=0}^{r} m_i u^i,$$

Where

$$m_i = \frac{F^{(i)}(a)}{i!}.$$

As a result, we obtain the following formula

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + P_r(u) + E_r(u)$$

by omitting the remainder we consider a new KPP equation with polynomial nonlinearities:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + P_r(u).$$

Therefore by means of (VIM) we get to the following correctional function

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left(\frac{\partial}{\partial \tau} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) - P_r\left(\tilde{u}_n(x,\tau)\right)\right) d\tau, \tag{5}$$

Making the above correction functional stationary, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\tau)\left(\frac{\partial}{\partial \tau} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) - P_r\left(\tilde{u}_n(x,\tau)\right)\right) d\tau,$$

Which yields the following stationary conditions:

$$1 + \lambda(t)|_{\tau=t} = 0,$$

$$\lambda'(\tau) = 0$$

The general Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = -1$$

By noting that $\delta \tilde{u}_n = 0$ and using $u_0(x, t)$ as a selective function, $u(x, t)$ will be obtained.

Numerical Examples:

Example 4.1:

We consider the following nonlinear Klein-Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u \ln u + u, \quad 0 < x < 0.1, \qquad t > 0. \tag{6}$$

subject to initial conditions:

$$u(x,0) = e^{\frac{1}{4} - \frac{1}{4}(1+x^2)},$$

$$u_t(x,0) = \frac{1}{2} e^{\frac{1}{4} - \frac{1}{4}(1+x^2)}$$

The exact solution is:

$$u(x,t) = e^{\frac{1}{4}(1+t^2) - \frac{1}{4}(1+x^2)}$$

To solve (6), we use iterative formula (4) to find the iteration for (6) given by:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \sinh(\tau - t)\left(\frac{\partial^2}{\partial \tau^2}u_n(x,\tau) - \frac{\partial^2}{\partial x^2}u_n(x,\tau) - P_2\big(u_n(x,\tau)\big)\right)d\tau$$

To get the iteration, we start with an initial approximation $u_0(x,0)=u(x,t)+tu_t(x,0)$ and we obtain the following successive approximations by considering $r=0$, as follows:

$$u_0(x,t) = e^{\frac{1}{4}-\frac{1}{4}(1+x^2)} + \frac{t}{2}e^{\frac{1}{4}-\frac{1}{4}(1+x^2)}$$

$$u_1(x,t) = e^{\frac{1}{4}(1+t^2)-\frac{1}{4}(1+x^2)} + g_1(x,t),$$

$$u_2(x,t) = e^{\frac{1}{4}(1+t^2)-\frac{1}{4}(1+x^2)} + g_2(x,t).$$

And so on.
Graphs of $g_1(x, t)$ and $g_2(x, t)$ are shown in Figure 1 and Figure 2 respectively.

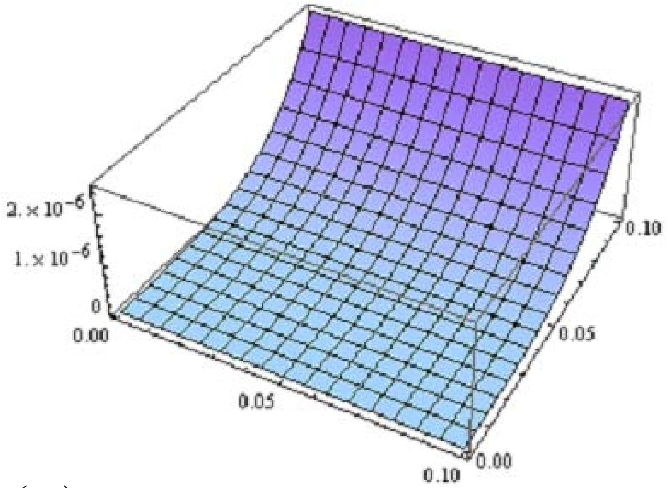


Fig. 1: The graph of $g_1(x, t)$.

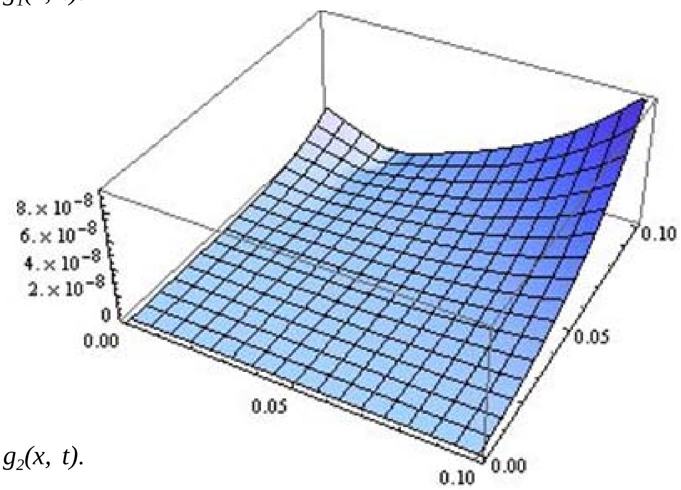


Fig. 2: The graph of $g_2(x, t)$.

Example 4.2:

We consider the following Kolmogorov-Petrovskii-Piskunov equation:

$$u_t = u_{xx} + 2u\ln u,\ 0 < x < 0.01,\ \ t > 0 \tag{7}$$

Subject to the initial condition:

$$u(x,0) = e^{3+2x}.$$

The exact functional separable solution is:

$$u(x,t) = \exp\left(2xe^{2t} + 2e^{4t} + e^{2t}\right)$$

To solve (7) we use iterative formula (5) to find the iteration for (7) given by:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial}{\partial \tau} u_n(x,\tau) - \frac{\partial^2}{\partial x^2} u_n(x,\tau) - 2u_n(x,\tau)P_2\left(u_n(x,\tau)\right)\right) d\tau.$$

To get the iteration, we start with an initial approximation $u_0(x,t)=e^{3+2x}$ and we obtain the following successive approximations by considering r=0, as follows:

$$u_0(x,t) = e^{3+2x}$$

$$u_1(x,t) = \exp\left(2xe^{2t} + 2e^{4t} + e^{2t}\right) + g_1(x,t),$$

$$u_2(x,t) = \exp\left(2xe^{2t} + 2e^{4t} + e^{2t}\right) + g_2(x,t).$$

And so on.
Graphs of $g_1(x, t)$ and $g_2(x, t)$ are shown in Figure 3 and Figure 4 respectively.

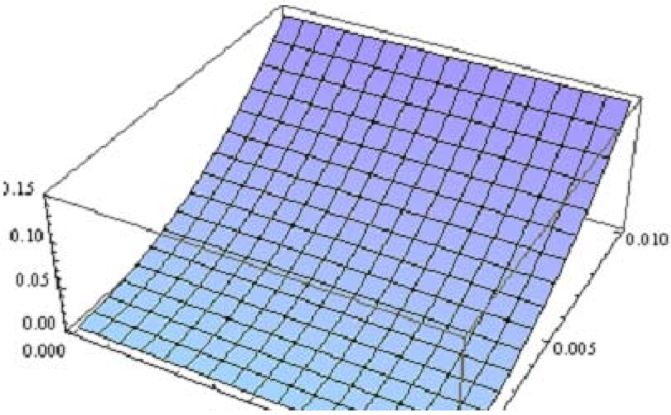
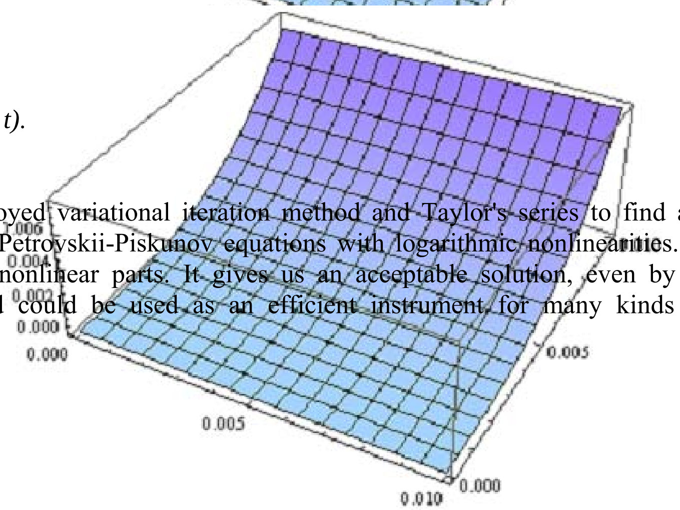


Fig. 3: The graph of $g_1(x, t)$.



Conclusion:

In this work we employed variational iteration method and Taylor's series to find a solution for Klein-Gordon and Kolmogorov-Petrovskii-Piskunov equations with logarithmic nonlinearities. We used r terms of Taylor's series instead of nonlinear parts. It gives us an acceptable solution, even by considering a small amount of r. This method could be used as an efficient instrument for many kinds of nonlinear partial differential equations.

Fig. 4: The graph of $g_2(x, t)$.

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