Australian Journal of Basic and Applied Sciences, 5(10): 1122-1129, 2011
ISSN 1991-8178

Relations on Weighted Quadratic Factor of Hardy and Bergman Spaces with a Sharp Estimate

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Abstract: We show construction of the translating of the inequalities, from the Hardy spaces to Bergman spaces with quadratic factors as weights and recover the original counterparts for Hardy spaces. We specify a relation of the equality on the general L^p-space, and compute a general case with a special sharp estimate for the extrema of the best possible constant.

Key wards: Hardy inequalities, Hardy-Littlewood inequalities, Fejer-Riesz inequality and Best possible constant.

INTRODUCTION

Let $D=\{z : |z|<1\}$ be the open unit disk in the complex plane *C*, and let $\partial D=\{z : |z|=1\}$ be the unit circle. Let $H(D)$ be the space of all analytic functions on the unit disk *D*. For *P* in $(0,\infty)$ the Hardy space H^p consists of analytical functions $f\in H(D)$ (f in D) such that $\|f\|_{H^p}^p=\sup_{0<p<r}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p\,dt<\infty$.

Each function f in H^p has a finite radial limit (Heakan Hedenmalm et al., 2000),(Peter and Alexander, 2004), which we denote by $f(\zeta)$, at almost every point ζ of the unit circle *T*. Furthermore, $\|f\|_{H^p}^p=\frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|^p\,dt=\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p\,dt$ for every f in H^P.

When $\alpha(\alpha+1)$ in $(-1,\infty)$ and *p* in $(0,\infty)$ the weighted Bergman space $A^p_{\alpha(\alpha+1)}$,(Heakan et al., 2000), (Kehe Zhu, 1990) Consists of analytic functions f in *H (D)* such that $\|f\|^p_{A^p_{\alpha(\alpha+1)}}=\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)<\infty$, where $dA_{\alpha(\alpha+1)}(z)=(\alpha^2+\alpha+1)(1-|z|^2)^{\alpha(\alpha+1)}dA(z)$ and dA is area measure on *D* normalized so that *A*(*D*)=1.

We now introduce the space $B_{\alpha(\alpha+1)}\{\alpha(\alpha+1)>0\}$ (David and Desmond, 2000) of functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ analytic in $|z|<1$, such that $\|f\|^2_{\alpha(\alpha+1)}=\sum_{n=0}^{\infty}(1+n)^{-\alpha(\alpha+1)}|a_n|^2<\infty$.

$B_{\alpha(\alpha+1)}$ is a Hilbert space with inner product $<f,g>=\sum_{n=0}^{\infty}(1+n)^{-\alpha(\alpha+1)}a_n\overline{b_n}$, where $g(z)=\sum_{n=0}^{\infty}b_nz^n$. Obviously, the polynomials are dense in $B_{\alpha(\alpha+1)}$, for each $\alpha(\alpha+1)>0$.

A straight forward calculations shows that

$$c_1\|f\|^2_{\alpha(\alpha+1)}\le\int_0^{2\pi}\int_0^1(1-r)^{\alpha^2+\alpha-1}|f(re^{i\theta})|^2\,dr\,d\theta\le c_2\|f\|^2_{\alpha(\alpha+1)} \tag{1}$$

For some positive constants c_1 and c_2 depending only on $\alpha(\alpha+1)$.

We have the following definition (see (David and Desmond, 2000)).

Definition 1.1:

Let *p* in $(1,\infty)$ and set $F(x)=\int_0^x f(t)\,dt$, where f is a non- negative measurable function on $(0,\infty)$. Then

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if $\varepsilon = \frac{1}{p'} = 1 - \frac{1}{p}$,

$$\int_0^\infty F^p(x)\, x^{p(\varepsilon-1)} dx \le c \int_0^\infty f^p(x)\, x^{\varepsilon p} dx \tag{2}$$

For some constant $c > 0$ independent of f. If $\varepsilon > \frac{1}{p'}$, the inequality takes the form

$$\int_0^\infty G^p(x)\, x^{p(\varepsilon-1)} dx \le c \int_0^\infty f^p(x)\, x^{\varepsilon p} dx \tag{3}$$

Where $G(x) = \int_0^\infty f(t)\,dt$. The best possible constants c in (2) and (3) are equal and this common value was determined by Landau as $c = \left|\varepsilon - \frac{1}{p'}\right|^{-p}$.

Relations on Inequalities of Hardy and Bergman Spaces:

Kehe Zhu (2004) consider the classical inequalities from (Peter Duren, 1970) to translate certain classical inequalities for Hardy spaces to inequalities for Bergman spaces, and then how to translate them back to the original inequalities for Hardy spaces.

The Fejer - Riesz inequality (Peter Duren, 1970).

Theorem 2.1:

Let p in $(0,\infty)$. Then $\int_{-1}^{1} |f(x)|^p dx \le \frac{1}{2}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta$ for all $f \in H^p$, moreover the constant $\frac{1}{2}$ is best possible for each p.

The second is an inequality of Hardy (Peter Duren, 1970).

Theorem 2.2:

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the Taylor series representation of a function in H^1, then $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi \|f\|_{H^1}$.

Moreover, the constant π is best possible.

If finding the best possible constant is a concern, then Hardy's inequality is a special case of the following inequality. The third due to Hardy- Littlewood (Peter Duren, 1970).

Theorem 2.3:

For each p in $(0,2]$ there exists a positive constant c_p such that $\sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p \le c_p \int_0^{2\pi} |f(e^{it})|^p dt$ for each function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^p.

Another theorem of Hardy-Littlewood (Peter and Alexander, 2004).

Theorem 2.4:

(i) If p in $(0,2)$, then $f \in H^p$ implies $\sum n^{p-2} |a_n|^p < \infty$ and $\{ \sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p \}^{\frac{1}{p}} \le c_p \|f\|_{H^{p}}$, where c_p denotes a constant depending only on p.

(ii) If p in $[2,\infty]$, then $\sum n^{p-2} |a_n|^p < \infty$ implies $f \in H^p$ and $\|f\|_{H^p} \le c_p \left(\sum_{n=0}^{\infty} (n+1)^{p-2} |a_n|^p \right)^{\frac{1}{p}}$.

Theorem 2.5:

Let p in $(1,\infty)$ and let $q = \frac{p}{p-1}$ be its conjugate exponent.

(i) If p in (1,2], then $f \in A^p$ implies that $\sum n^{1-q} |a_n|^q < \infty$, and $\left(\sum_{n=0}^{\infty} \frac{|a_n|^q}{(n+1)^{q-1}}\right)^{\frac{1}{q}} \le \|f\|_p$.

(ii) If p in $[2,\infty)$, then $\sum n^{1-q}|a_n|^q < \infty$ implies that $f \in A^p$; and $\|f\|_p \le (\sum_{n=0}^{\infty} \frac{|a_n|^q}{(n+1)^{q-1}})^{\frac{1}{q}}$

We begin with the weighted quadratic factor Bergman space version of the Fejer-Riesz inequality.

Theorem 2.6:

Suppose that p in $(0,\infty)$ and $\alpha(\alpha+1)$ in $(-1,\infty)$. Then $\int_{-1}^{1}(1-|x|)^{\alpha^2+\alpha+1}|f(x)|^p\,dx \le \pi\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z)$ for all f in $A^p_{\alpha(\alpha+1)}$.

Proof:

Let f be a function in $A^p_{\alpha(\alpha+1)}$. For each r satisfying $0 \le r < 1$ the function f_r given by $z \to f(rz)$ is clearly in H^p, so by Theorem 2.1.

$$\int_{-1}^{1}|f(rx)|^p\,dx \le \frac{1}{2}\int_0^{2\pi}|f(re^{it})|^p\,dt \cdot \tag{4}$$

We multiply both sides of (4) by $\frac{1}{\pi}(\alpha^2+\alpha+1)r(1-r^2)^{\alpha(\alpha+1)}dr$, and integrate from 0 to 1 to obtain $\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_{-1}^{1}|f(rx)|^p dxdr \le \frac{\alpha^2+\alpha+1}{2\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_0^{2\pi}|f(re^{it})|^p dtdr \cdot$

A simple change of variables transforms the left hand side to $rx = y$ which implies that.

$$\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_{-r}^{r}\frac{|f(y)|^p}{r}dydr = \frac{\alpha^2+\alpha+1}{\pi}\int_0^1 (1-r^2)^{\alpha(\alpha+1)}.\int_{-r}^{r}|f(x)|^p dxdr$$

we then use Fubini's theorem to rewrite this as:

$$\frac{\alpha^2+\alpha+1}{\pi}\int_{-1}^{1}|f(x)|^p\,dx\int_{|x|}^{1}(1-r^2)^{\alpha(\alpha+1)}dr \cdot$$

For the right hand side we have:

$$\frac{\alpha^2+\alpha+1}{2\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}|f(re^{it})|^p\,dtdr\,.$$

Let $z = re^{it}$ then $dA_{\alpha(\alpha+1)}(z) = \frac{\alpha^2+\alpha+1}{\pi}r(1-r^2)^{\alpha(\alpha+1)}drdt$

$$|f(z)|^p\,dA_{\alpha(\alpha+1)}(z) = \frac{\alpha^2+\alpha+1}{\pi}r(1-r^2)^{\alpha(\alpha+1)}|f(z)|^p\,drdt$$

$$\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z) = \frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_0^{2\pi}|f(x)|^p dtdr$$

$$\frac{1}{2}\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z) = \frac{\alpha^2+\alpha+1}{2\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_0^{2\pi}|f(x)|^p dtdr \cdot$$

From left and right hand sides we have

$$\frac{\alpha^2+\alpha+1}{\pi}\int_{-1}^{1}|f(z)|^p\int_{|x|}^{1}(1-r^2)^{\alpha(\alpha+1)}drdx \le \frac{1}{2}\int_D |f(x)|^p dA_{\alpha(\alpha+1)}(z) \cdot$$

Since $\frac{\pi}{2}\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z) \ge (\alpha^2+\alpha+1)\int_{-1}^{1}|f(x)|^p\int_{|x|}^{1} r(1-r^2)^{\alpha(\alpha+1)}drdx$

and since $(\alpha^2+\alpha+1)\int_{|x|}^{1}(1-r^2)^{\alpha(\alpha+1)}(-r)dr=\frac{1}{2}(1-|x|)^{\alpha^2+\alpha+1}$,

then we have $\frac{\pi}{2}\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z)\geq\int_{-1}^{1}\frac{1}{2}(1-|x|)^{\alpha^2+\alpha+1}|f(x)|^p\,dx$

we conclude that $\int_{-1}^{1}(1-|x|)^{\alpha^2+\alpha+1}|f(x)|^p\,dx\leq\pi\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)$, which completing the proof of the Theorem. We are not certain that the constant π in Theorem 2.6 is best possible for any fixed $\alpha(\alpha+1)$ and p. However, because the constant $\frac{1}{2}$ in Theorem2.1 is sharp.

The counterpart of weighted quadratic factor Hardy's inequality for weighted quadratic factor Bergman spaces is expressed by the next result.

Theorem 2.7:

Suppose that $\alpha(\alpha+1)$ in $(-1,\infty)$ and that $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is a function in $A^1_{\alpha(\alpha+1)}$.

Then $\sum_{n=0}^{\infty}\frac{\Gamma(\alpha^2+\alpha+1)\Gamma(\frac{n}{2}+1)}{(n+1)\Gamma(\frac{n}{2}+\alpha^2+\alpha+2)}|a_n|\leq\pi\int_D|f(z)|dA_{\alpha(\alpha+1)}(z)$.

Proof:

For fixed r in $[0,1)$ we first apply Hardy's inequality to the dilated function f_r, which has the Taylor expansion $f_r(z)=f(rz)=\sum_{n=1}^{\infty}a_n r^n z^n$, $\int_{-1}^{1}|f(rx)|^p\,dx\leq\frac{1}{2}\int_0^{2\pi}|f(re^{it})|^p\,dt$

$$\int_{-1}^{1}\left|\sum_{n=1}^{\infty}a_n r^n x^n\right|^p dx\leq\frac{1}{2}\int_0^{2\pi}|f(z)|^p\,dt \tag{5}$$

And then integrate the resulting inequality with respect to the measure $(\alpha^2+\alpha+1)r(1-r^2)^{\alpha(\alpha+1)}dr$ on the interval $[0,1)$.

The right hand side implies that $\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)=\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}|f(z)|^p\,dtdr$.

For $p=1$, $z=re^{it}$, $f(z)=\sum a_n r^n z^n$ then

$$\frac{\pi}{2}\int_D|f(z)|dA_{\alpha(\alpha+1)}(z)=\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}\left|\sum a_n r^n e^{\text{int}}\right|dtdr. \tag{6}$$

And the left hand side shows that by letting $r^2=s$

$$\sum_{n=1}^{\infty}\frac{\alpha^2+\alpha+1}{2}\int_0^1(1-s)^{\alpha(\alpha+1)}s^{\frac{n}{2}}ds\,|a_n|\left|\int_{-1}^{1}|x|\,dx\right| = \frac{\alpha^2+\alpha+1}{2}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha^2+\alpha+1)\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\alpha^2+\alpha+2)}|a_n|\left|\int_{-1}^{1}|x|^n\,dx\right. \tag{7}$$

From (6)and(7) we have

$$\frac{\alpha^2+\alpha+1}{2}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha^2+\alpha+1)\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\alpha^2+\alpha+2)}|a_n|\left|\int_{-1}^{1}|x|^n\,dx\right.\leq\pi\int_D|f(z)|dA_{\alpha(\alpha+1)}(z).$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\Gamma(\alpha^2+\alpha+2)\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\alpha^2+\alpha+2)}|a_n|\int_{-1}^{1}|x|^n\,dx\le\pi\int_D|f(z)|dA_{\alpha(\alpha+1)}(z)$$

The Hardy-Littlewood inequality also has its analogue in the weighted quadratic factor Bergman space setting.

Theorem 2.8:

Suppose that p in $(0,2]$ and $\alpha(\alpha+1)$ in $(-1,\infty)$. If c_p is the constant from Theorem2.3, then for each function $f(z)=\sum_{n=0}^{\infty}a_nz^n$ in $A^p_{\alpha(\alpha+1)}$, it is the case that

$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha^2+\alpha+2)\Gamma(\frac{np}{2}+1)}{\Gamma(\frac{np}{2}+\alpha^2+\alpha+2)}(n+1)^{p-2}|a_n|^p\le 2\pi\,c_p\int_D|f(z)|dA_{\alpha(\alpha+1)}(z).$$

Proof:

For fixed r in $[0,1)$ the dilated function f_r, which has the Taylor expansion we have

$$\int_{-1}^{1}|f(rx)|^p\,dx\le\frac{1}{2}\int_0^{2\pi}|f(re^{it})|^p\,dt$$

$$\int_{-1}^{1}\left|\sum_{n=1}^{\infty}a_nr^nx^n\right|^p dx\le\frac{1}{2}\int_0^{2\pi}|f(z)|^p\,dt \tag{8}$$

Then integrate the resulting inequality with respect to the measure $(\alpha^2+\alpha+1)r(1-r^2)^{\alpha(\alpha+1)}dr$ on the interval $[0,1)$ .We have for the right hand side

$$\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)=\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}|f(z)|^p\,dtdr.$$

$$\frac{\pi}{2}\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)=\frac{\alpha^2+\alpha+1}{2}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}.\int_0^{2\pi}\left|\sum a_nr^ne^{\text{int}}\right|^p dtdr \tag{9}$$

For the left hand side we have by letting $r^2=s$

$$\left|\sum_{n=1}^{\infty}\frac{\alpha^2+\alpha+1}{2}\int_0^1(1-s)^{\alpha(\alpha+1)}s^{\frac{np}{2}}ds\int_{-1}^{1}(a_n)^px^{np}dx\right|=\left|\frac{\alpha^2+\alpha+1}{2}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha^2+\alpha+1)\Gamma(\frac{np}{2}+1)}{\Gamma(\frac{np}{2}+\alpha^2+\alpha+2)}\int_{-1}^{1}(a_n)^px^{np}dx\right| \tag{10}$$

from (9) and (10) we conclude that

$$\left|(\alpha^2+\alpha+1)\sum_{n=1}^{\infty}\frac{\Gamma(\alpha^2+\alpha+1)\Gamma(\frac{np}{2}+1)}{\Gamma(\frac{np}{2}+\alpha^2+\alpha+2)}\int_{-1}^{1}(a_n)^px^{np}dx\right|\le\pi\int_D|f(z)|^p\,dA_{\alpha(\alpha+1)}(z)$$

Therefore $\sum_{n=0}^{\infty}\frac{\Gamma(\alpha^2+\alpha+2)\Gamma(\frac{np}{2}+1)}{\Gamma(\frac{np}{2}+\alpha^2+\alpha+2)}(n+1)^{p-2}|a_n|^p\le 2\pi\,c_p\int_D|f(z)|dA_{\alpha(\alpha+1)}(z).$

Corollary 2.9:

Suppose that p in $(0,2]$ and $\alpha(\alpha+1)$ in $(-1,\infty)$. Then there exists a constant $c>0$ (depending on $\alpha(\alpha+1)$ and p) such that

$$\sum_{n=0}^{\infty}(n+1)^{p-\alpha^2-\alpha-3}\left|a_n\right|^p \le c\int_D \left|f(z)\right|^p dA_{\alpha(\alpha+1)}(z) \tag{11}$$

for every function $f(z)=\sum_{n=0}^{\infty}a_n z^n$ in $A^p_{\alpha(\alpha+1)}$.

We now show how the inequalities for weighted quadratic factor Bergman spaces can be used to recover the original counterparts for Hardy spaces.

Proposition 2.10:

Suppose that p in $(0,\infty)$ and that f is in H^p.Then f belongs to $A^p_{\alpha(\alpha+1)}$ for every $\alpha(\alpha+1)$ in $(-1,\infty)$. Moreover, $\lim_{\alpha(\alpha+1)\to -\overset{+}{1}}\|f\|_{A^p_{\alpha(\alpha+1)}}=\|f\|_{H^p}$.

Proof:

By switching to polar coordinates, we find that

$$\|f\|^p_{A^p_{\alpha(\alpha+1)}}=(\alpha^2+\alpha+1)\int_D \left|f(z)\right|^p(1-|z|^2)^{\alpha(\alpha+1)}dA(z)$$

$$=\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}\left|f(re^{it})\right|^p dtdr$$

$$\le 2(\alpha^2+\alpha+1)\|f\|^p_{H^p}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}dr=\|f\|^p_{H^p}.$$

This shows that f belongs to $A^p_{\alpha(\alpha+1)}$ for all $\alpha(\alpha+1)$ in $(-1,\infty)$ and that

$$\lim_{\alpha(\alpha+1)\to -\overset{+}{1}}\sup\|f\|_{A^p_{\alpha(\alpha+1)}}\le\|f\|_{H^p} \tag{12}$$

on the other hand, for any $\varepsilon>0$ there exists some σ in $(0,1)$ such that $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{it})\right|^p dt>\|f\|^p_{H^p}-\varepsilon$ for all r in $(\sigma,1)$.

It follows that $\|f\|^p_{A^p_{\alpha(\alpha+1)}}=\frac{\alpha^2+\alpha+1}{\pi}\int_0^1 r(1-r^2)^{\alpha(\alpha+1)}\int_0^{2\pi}\left|f(re^{it})\right|^p dtdr$

$$>2(\alpha^2+\alpha+1)(\|f\|^p_{H^p}-\varepsilon)\int_\sigma^1 r(1-r^2)^{\alpha(\alpha+1)}dr$$

$$=(\|f\|^p_{H^p}-\varepsilon)(1-\sigma^2)^{\alpha^2+\alpha+1}.$$

Letting $\alpha(\alpha+1)\to-\overset{+}{1}$, we obtain $\lim_{\alpha(\alpha+1)\to -\overset{+}{1}}\inf\|f\|^p_{A^p_{\alpha(\alpha+1)}}\ge\|f\|^p_{H^p}-\varepsilon$. Since ε is arbitrary, we must have

$$\lim_{\alpha(\alpha+1)\to -\overset{+}{1}}\inf\|f\|_{A^p_{\alpha(\alpha+1)}}\ge\|f\|_{H^p}. \tag{13}$$

In combination with (12) and (13) completes the proof of the proposition.

The Main Results:

Here we show the results of various relations between weighted quadratic factor Bergman spaces and Hardy spaces followed by a sharp estimate of an extrema of the best possible constant.

Theorem 3.1:

Suppose that p in $(0,\infty]$ and $\alpha(\alpha+1)$ in $(-1,\infty]$, then

(i) $\sum_{n=0}^{\infty}(n+1)^{p-2}|a_n|^p \le K^p_{\alpha(\alpha+1)}\|f\|^p_{A^p_{\alpha(\alpha+1)}}$ for $K^p_{\alpha(\alpha+1)}$, a constant depending on $\alpha(\alpha+1)$ and p and every function $f(z)=\sum_{n=0}^{\infty}a_n z^n$ in $A^p_{\alpha(\alpha+1)}$, Furthermore

(ii) $\|f\|_{H^p} \le C^p_{\alpha(\alpha+1)}\|f\|_{A^p_{\alpha(\alpha+1)}}$ for $C^p_{\alpha(\alpha+1)}$, a constant depending on $\alpha(\alpha+1)$ and p

(iii) $\|f\|_{H^p} = C'\,\|f\|_{A^p_{\alpha(\alpha+1)}}$ where C' depending on $\alpha(\alpha+1)$ and p

(iv) $\|f\|_{A^p_{\alpha(\alpha+1)}} \le K^p_{\alpha(\alpha+1)}\|f\|_{H^p}$

Proof:

For $\alpha(\alpha+1)$ in $(-1,\infty)$, p in $(0,2]$ and $f\in H^p$

(i) For

$$\sum_{n=0}^{\infty}(n+1)^{p-\alpha(\alpha+1)-3}|a_n|^p = \sum_{n=0}^{\infty}(n+1)^{p-2}|a_n|^p\,(n+1)^{-\alpha(\alpha+1)-1}, \tag{14}$$

Then using Holder inequality and Corollary2.9, if p in $(0,2)$ then we have from (11) and (14) that

$$\sum_{n=0}^{\infty}(n+1)^{p-2}|a_n|^p \le K^p_{\alpha(\alpha+1)}\|f\|^p_{A^p_{\alpha(\alpha+1)}}$$

where, $K^p_{\alpha(\alpha+1)} = \frac{c}{r^{\alpha(\alpha+1)}}$,(c depend on $\alpha(\alpha+1)$ and p) and $r^p_{\alpha(\alpha+1)} = \sum_{n=0}^{\infty}\left\{\left|\frac{1}{(n+1)^{\alpha^2+\alpha+1}}\right|^p\right\}^{\frac{1}{p}}$

(ii) If p in $[0,\infty]$ then using Theorem2.4 (ii) we have,

$$\|f\|_{H^p} \le c_p\left(\sum_{n=0}^{\infty}(n+1)^{p-\alpha(\alpha+1)-3}|a_n|^p\right)^{\frac{1}{p}} \tag{15}$$

Hence from (15) and (i) we have $\|f\|_{H^p} \le c^p_{\alpha(\alpha+1)}\|f\|_{A^p_{\alpha(\alpha+1)}}$, where $c^p_{\alpha(\alpha+1)} = c_p\left(K^p_{\alpha(\alpha+1)}\right)^{\frac{1}{p}}$

(iii) From the first part of the proof of Proposition2.10 we can find, for a suitable constant C' which depend on $\alpha(\alpha+1)$ and p , that

$$\|f\|_{H^p} = C'\,\|f\|_{A^p_{\alpha(\alpha+1)}}$$

(iv) It is easily find, from Proposition2.10, that $\|f\|_{A^p_{\alpha(\alpha+1)}} \le K^p_{\alpha(\alpha+1)}\|f\|_{H^p}$ for a suitable constant $K^p_{\alpha(\alpha+1)} \ge 0$.

Theorem 3.2:

Suppose that p in $(0,\infty)$, b in $(-1,\infty)$, $0<a<\infty$ and $\alpha(\alpha+1)$ in $(-1,\infty)$, for the extrema of a best possible c_p with a π-area of $A^p_{\alpha(\alpha+1)}$ we have $c_p \ge 0.5998$.

Proof:

For(see Definition1.1)

$$\int_{-1}^{1}(1-|x|)^{\alpha^2+\alpha+1}|f(x)|^p\,dx \le c_p\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z)\text{, and } f\in A^p_{\alpha(\alpha+1)}.$$

Let $f(x)=(1-x)^{a\,\alpha(\alpha+1)+b}$, Theorem 2.6 gives

$$\int_{-1}^{1}(1-x)^{\alpha^2+\alpha+1}\left|(1-x)^{a\alpha(\alpha+1)+b}\right|^p dx = \int_{-1}^{1}(1-x)^{k-1}dx = \frac{2^k}{k}\text{, where}$$

$k = \alpha(\alpha+1)(1+a\ p)+(2+b\ p)$, and $c_p \geq \frac{2^k}{\pi\,\mathrm{k}}$.

By using simple computations, and setting $\alpha(\alpha+1)$ instead of α, we have for the extrema (see the figure) that $\Phi_k = \frac{2^k}{\pi\,\mathrm{k}} = \frac{2^{\frac{1}{\ln 2}}}{\pi}.\ln 2 = 0.5998$ at $k = \frac{1}{\ln 2} = 1.4427$

Remark 3.3:

For example, in Theorem3.2, if we take, $\overline{f_{\alpha(\alpha+1)}(z)} = \frac{z-\alpha(\alpha+1)}{1-\overline{\alpha}(\overline{\alpha}+1)\,\mathrm{z}}$, and for $p>0$, $a=b=1$ then we can show that $\int_{-1}^{1}(1-|x|)^{\alpha^2+\alpha+1}|f(x)|^p\,dx = \frac{8}{3}$, and $\int_D |f(z)|^p\,dA_{\alpha(\alpha+1)}(z) = \int_0^{2\pi}\frac{z-\alpha(\alpha+1)}{1-\overline{\alpha}(\overline{\alpha}+1)\,z}(\alpha^2+\alpha+1)(1-|z|^2)^{\alpha(\alpha+1)}dA(z) = 2\pi^2$.

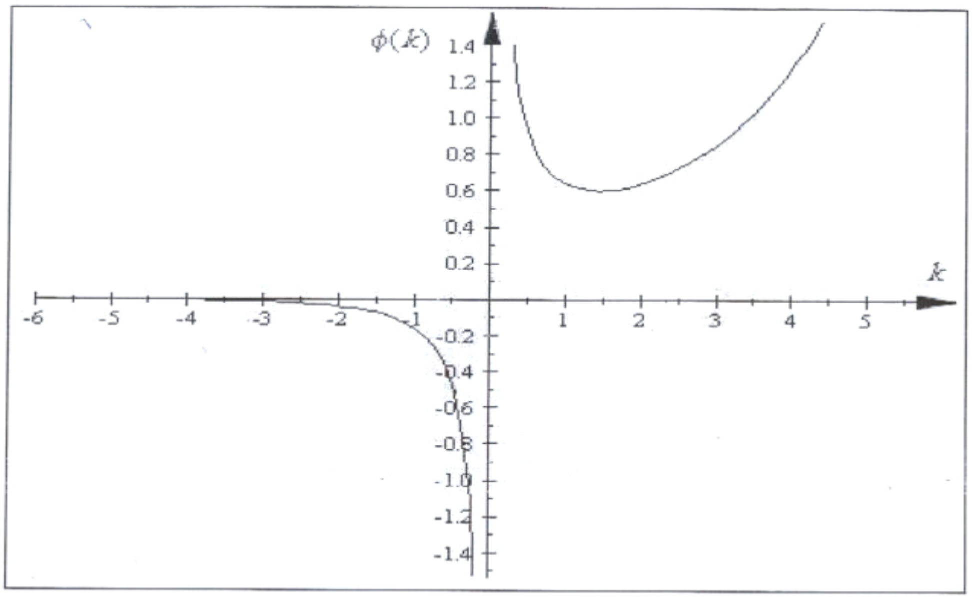

Fig. 1: Graph of the fucntion: $\Phi(k) = \frac{2^k}{\mathrm{xk}}$

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