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Numerical Integration of a Full Fuzzy Riemann Double Integral

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Abstract: In this paper, the numerical integration of a full fuzzy Riemann double integral is proposed. At first, we introduce a fuzzy Riemann double integral whose integrand is a fuzzy function and also the limits of integration are fuzzy numbers. In this case, we prove a theorem to show the α-level set of the fuzzy double integral which is a closed interval where end points are double crisp Riemann integrals. In this case, we apply the double Simpson's rule in order to approximate these double integrals. Also, we present an algorithm to approximate the value of the membership function of the double fuzzy integral in a given point like r in 0-level. Finally, two numerical examples are solved to illustrate the efficiency of the proposed method.

Key words: Full fuzzy double integrals, Fuzzy functions, Fuzzy numbers, Double Simpson's rule, α-level sets.

INTRODUCTION

The concept of fuzzy integral was introduced by Sugeno (Sugeno, 1974). After that, some formulations of fuzzy integrals have been developed (Sims, 1990). Since, finding the value of a fuzzy integral is complicated and difficult analytically, in recent years some numerical methods have been proposed for approximating a fuzzy definite integral. In (Allahviranloo, 2005; Allahviranloo, 2005; Allahviranloo, 2006; Allahviranloo, 2005), by using the parametric form of a fuzzy number and by applying numerical integration methods, a fuzzy integral with fuzzy-valued function over a crisp interval has been solved. The integral of a fuzzy-valued function has been evaluated from an analytical scheme by introducing the Henstock integrals and applying the quadrature rules in (Bede, 2004; Gong Z., 2004). In (Fariborzi Araghi, 2004; Fariborzi Araghi, 2006), by using the concept of α-level of a fuzzy number and applying the Newton-Cotes integration method, a fuzzy definite integral with a fuzzy function over a fuzzy interval has been estimated. In this case, the value of membership function for a fuzzy integral is approximated in a given point like r in 0-level by introducing a numerical algorithm based on the Wu's method which has been mentioned in (Wu Hsein-Chung, 2000).

In this paper, we develop this work for a fuzzy Riemann double integral as follows:

$$\tilde{I} = \int_{\tilde{a}}^{\tilde{b}} \int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}, \tag{1}$$

where $\tilde{f}$ is a fuzzy-valued function and $\tilde{a},\tilde{b},\tilde{c}$ and $\tilde{d}$ are fuzzy numbers. In order to approximate $\tilde{I}$, at first we introduce the α-level of this fuzzy number as $I_\alpha = [\underline{I}_\alpha, \overline{I}_\alpha]$, $0 \le \alpha \le 1$, where $\underline{I}_\alpha$, $\overline{I}_\alpha$ are the end points of I_α. Let $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ be the end-points of the α-cut set of the fuzzy functions $\tilde{f}$. Then, by using the double Simpson's rule, we can estimate the crisp double integrals, $\underline{I}_\alpha$ and $\overline{I}_\alpha$ for all $\alpha \in [0,1]$. Also, we propose an algorithm in order to estimate the value of membership function of $\tilde{I}$, $\mu_{\tilde{I}}(r)$, in a given point $r \in I_0$. In Section 2, we present some basic definitions and properties of fuzzy sets and fuzzy numbers

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and also some basic theorems which are used in the work. In Section 3, we introduce the full Riemann fuzzy double integral and its α-level set. The computational method and numerical algorithm will be discussed in section 4. Also, two numerical examples are solved to show the efficiency of the given algorithm.

2 Preliminaries:

We start this section giving some properties of fuzzy sets and operations of fuzzy numbers,which will be used in the rest of this paper.

Definition 2.1:

Let X be a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$. We can also write the fuzzy set $\tilde{A}$ as $\{(x, \mu_{\tilde{A}}) : x \in X\}$. We denote $A_\alpha = \{x : \mu_{\tilde{A}} \geq \alpha\}$ as the α- level set of $\tilde{A}$, where $\tilde{A}_0$ is the closure of the set $\{x : \mu_{\tilde{A}} \neq o\}$

Definition 2.2:

(i) $\tilde{A}$ is called normal fuzzy set if there exists x such that $\mu_{\tilde{A}}(x) = 1$.

(ii) $\tilde{A}$ is called convex fuzzy set if $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$, for $0 \leq \lambda \leq 1$.

Proposition 2.1:

(Zadeh, L.A., 1975). $\tilde{A}$ is a convex fuzzy set if and only if $\{x : \mu_{\tilde{A}} \geq \alpha\}$ is a convex set for all $0 \leq \alpha \leq 1$.

We can develop the definition of an upper semicountinous function mentioned in (Rudin, W., 1986) for a two-variables function as follows.

Definition 2.3:

Let $f(x, y)$ be a real-valued function on a topological space. If $\{(x, y) : f(x, y) \geq \alpha\}$ is closed for all α, $f(x, y)$ is said to be upper semicontinuous.

Definition 2.4:

(i) $\tilde{a}$ is called fuzzy number if $\tilde{a}$ is a normal convex fuzzy set and the α-level set, $\tilde{a}_\alpha$, is bounded all $\alpha \neq 0$. The set of all fuzzy numbers is denoted by *F*.

(ii) $\tilde{a}$ is called closed fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function $\mu_{\tilde{a}}$ is upper semicontinuous. The set of all closed fuzzy numbers is denoted by F_{cl}.

(iii) $\tilde{a}$ is called bounded fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function $\mu_{\tilde{a}}$ has compact support. The set of all bounded fuzzy numbers is denoted by F_b .

Remark 2.1:

Let $\tilde{a}$ be a fuzzy number. We regard, $\tilde{a}_0$, the 0-level set of $\tilde{a}$ as the closure of the set $\{(x, y) : \mu_{\tilde{a}}(x, y) \neq 0\}$. If $\tilde{a}$ is a bounded fuzzy number then $\tilde{a}_0$ is a compact set.

Proposition 2.2:
If $\tilde{a}$ is a closed fuzzy number then the α-level set of $\tilde{a}$ is a closed interval, which is denoted by $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$.

Proof:

$\mu_{\tilde{a}}$ is an upper semicontinuous function, so the α-level set $\tilde{a}_\alpha = \{(x,y) : \mu_{\tilde{a}}(x,y) \geq \alpha\}$ is a closed set. Since $\tilde{a}$ is a convex fuzzy set, by proposition 2.1, $\tilde{a}_\alpha$ is a closed interval.

Definition 2.5:

Let $\tilde{a}$ be a fuzzy number.

(i) $\tilde{a}$ is called nonnegative fuzzy number if $\mu_{\tilde{a}}(x) = 0, \forall x < 0$.

(ii) $\tilde{a}$ is called nonpositive fuzzy number if $\mu_{\tilde{a}}(x) = 0, \forall x > 0$.

(iii) $\tilde{a}$ is called positive fuzzy number if $\mu_{\tilde{a}}(x) = 0, \forall x \leq 0$.

(iv) $\tilde{a}$ is called negative fuzzy number if $\mu_{\tilde{a}}(x) = 0, \forall x \geq 0$.

Definition 2.6:

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers.

(i) The membership function of $\tilde{a}.\tilde{b}$ is defined by

$$\mu_{\tilde{a}.\tilde{b}}(z) = \sup_{xoy=z} \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\}. \tag{2}$$

(ii)The membership function of the inverse of $\tilde{a}$ is defined by

$$\mu_{1/\tilde{a}}(z) = \sup_{z=1/x, x\neq 0} \min \mu_{\tilde{a}}(x) = \mu_{\tilde{a}}(1/z). \tag{3}$$

(iii)The quotient of $\tilde{a}$ and $\tilde{b}$ is defined by

$$\tilde{a}/\tilde{b} = \tilde{a} \otimes (1/\tilde{b}). \tag{4}$$

Proposition 2.3:
(Wu Hsein-Chung, 2000). Let $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers.

(i) $\tilde{a} \oplus \tilde{b}$, $\tilde{a}/\tilde{b}$ and $\tilde{a} \otimes \tilde{b}$ are also closed fuzzy numbers.

(ii) If $\tilde{b}$ is positive or negative then $\tilde{a}/\tilde{b}$ is also a closed fuzzy number.

Proposition 2.4:
(Wu Hsein-Chung, 2000). If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy numbers then

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U], \tag{5}$$

$$(\tilde{a} - b)_\alpha = [\tilde{a}_\alpha^L - \tilde{b}_\alpha^U, \tilde{a}_\alpha^U - \tilde{b}_\alpha^L], \tag{6}$$

$$(\tilde{a}\otimes\tilde{b})_\alpha=[\min\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L,\tilde{a}_\alpha^L\tilde{b}_\alpha^U,\tilde{a}_\alpha^U\tilde{b}_\alpha^L,\tilde{a}_\alpha^U\tilde{b}_\alpha^U\},\max\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L,\tilde{a}_\alpha^L\tilde{b}_\alpha^U,\tilde{a}_\alpha^U\tilde{b}_\alpha^L,\tilde{a}_\alpha^U\tilde{b}_\alpha^U\}]. \quad (7)$$

If $\tilde{a}$ and $\tilde{b}$ are two nonnegative closed fuzzy numbers then

$$(\tilde{a}\otimes\tilde{b})_\alpha=[\tilde{a}_\alpha^L\tilde{b}_\alpha^L,\tilde{a}_\alpha^U\tilde{b}_\alpha^U]. \quad (8)$$

Definition 2.7:

(i) Let $[a, b]$ and $[c, d]$ be closed intervals. We say that $[a,b]\succ_{\text{int}}[c,d]$ if and only if $a\geq c$ and $b\geq d$.

(ii) Let $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers. We say that $\tilde{a}\succ\tilde{b}$ if and only if $\tilde{a}_\alpha\succ_{\text{int}}\tilde{b}_\alpha$ for all $\alpha\in[0,1]$.

(iii) Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. We say that $\tilde{a}$ is equal to $\tilde{b}$, denoted as $\tilde{a}=\tilde{b}$, if and only if $\tilde{a}_\alpha=\tilde{b}_\alpha$ for all $\alpha\in[0,1]$.

Proposition 2.5:

(Wu Hsein-Chung, 2000). "$\succ$ " is a partial ordering on F_{cl} .

Definition 2.8:

We say that $\tilde{a}$ is a crisp number with value m if and only if its membership function is

$$\mu_{\tilde{a}}(r)=\begin{cases}1 & if r=m,\\ 0 & otherwise.\end{cases} \quad (9)$$

Let $\tilde{a}$ be a fuzzy number. We define the membership functions of $\tilde{a}^+$ and $\tilde{a}^-$ as

$$\mu_{\tilde{a}^-}(r)=\begin{cases}\mu_{\tilde{a}}(r) & if r<0,\\ 1 & if r=0 \quad and \quad \mu_{\tilde{a}}(r)<1,\forall r<0,\\ \mu_{\tilde{a}}(0) & if r=0\ and \quad \exists r<0\ such\ that\ \mu_{\tilde{a}}(r)=1,\\ 0 & otherwise.\end{cases} \quad (10)$$

and

$$\mu_{\tilde{a}^+}(r)=\begin{cases}\mu_{\tilde{a}}(r) & if r>0,\\ 1 & if r=0 \quad and \quad \mu_{\tilde{a}}(r)<1,\forall r>0,\\ \mu_{\tilde{a}}(0) & if r=0and \quad \exists r>0\ such\ that\ \mu_{\tilde{a}}(r)=1,\\ 0 & otherwise.\end{cases} \quad (11)$$

Remark 2.2:

(i) $\tilde{a}^+$ is a nonnegative fuzzy number and $\tilde{a}^-$ is a nonpositive fuzzy number.

(ii) If $\tilde{a}$ is a closed fuzzy number then $\tilde{a}^+$ and $\tilde{a}^-$ are also closed fuzzy numbers.($\tilde{a}_\alpha^+$ and $\tilde{a}_\alpha^-$ are closed intervals for all α, thus their membership function $\mu_{\tilde{a}^+}(r)$ and $\mu_{\tilde{a}^-}(r)$ are upper semicontinuous from Definition 2.3). Furthermore, we have $\tilde{a}_\alpha=\tilde{a}_\alpha^+\oplus_{int}\tilde{a}_\alpha^-=(\tilde{a}^+\oplus\tilde{a}^-)_\alpha$ for all α (by proposition 2.4). Thus $\tilde{a}=\tilde{a}^+\oplus\tilde{a}^-$. We call $\tilde{a}^+$ as the positive part of $\tilde{a}$ and $\tilde{a}^-$ as the negative part of $\tilde{a}$.

Proposition 2.6:
(i) (Resolution Identity, Zadeh, 1975). Let $\tilde{A}$ be a fuzzy set with membership function $\mu_{\tilde{A}}$ and $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha\}$. Then

$$\mu_{\tilde{A}}(x) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{A_\alpha}(x). \tag{12}$$

(ii) (Negoita and Ralescu, 1975) Let A be a set and $\{A_\alpha : \alpha \in [0,1]\}$ be a family of A such that

(a) $A_0 = A$,

(b) $A_\beta \subseteq A_\alpha$ for $\alpha < \beta$,

(c) $A_\alpha = \cap_{n=1}^{\infty} A_{\alpha_n}$ for $\alpha_n \uparrow \alpha$,

then the function $\mu : A \to [0,1]$ defined by

$$\mu(x) = \sup_{0 \leq \alpha \leq 1} \alpha 1_{A_\alpha}(x), \tag{13}$$

has the property that

$A_\alpha = \{x : \mu(x) \geq \alpha\}$ for all $\alpha \in [0,1]$.

Let $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \leq \alpha \leq 1\}$ be a family of closed intervals. Then, we can induce a fuzzy set $\tilde{A}$ with membership function (12).

We say that $\{A_\alpha\}$ is decreasing with respect to α. if $A_\beta \subseteq A_\alpha$ for $\alpha < \beta$.

Proposition 2.7:
(Bazarra and Shetty 1993). Let S be a compact set in R^n. If *f* is upper semicontinuous on *S* then *f* assumes maximum over *S*, and if *f* is lower semicontinuous on *S* then *f* assumes minimum over *S*.

Proposition 2.8:
(Wu Hsein-Chung, 2000). Let $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \leq \alpha \leq 1\}$ be a family of closed intervals. If $A_1 \neq 0$ and $\tilde{A}$ is induced by $\{A_\alpha\}$, then $\tilde{A}$ is a normal fuzzy set.

Proposition 2.9:
(Wu Hsein-Chung, 2000). (i) Let $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \leq \alpha \leq 1\}$ be decreasing with respect to α. Then $f(\alpha) = \alpha 1_{A_\alpha}(r)$ is upper semicontinuous for any fixed *r*.

(ii) If $\sup_{0 \leq \alpha \leq 1} \alpha 1_{A_\alpha}(r) \geq \beta$ then $\exists \alpha_0 \geq \beta$ such that $r \in A_{\alpha_0}$.

Proposition 2.10:
(Wu Hsein-Chung, 2000). If $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \leq \alpha \leq 1\}$ is decreasing with respect to α, then the fuzzy set $\tilde{A}$ induced by $\{A_\alpha\}$ is a convex fuzzy set.

Proposition 2.11:
(Wu Hsein-Chung, 2000). Let $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \leq \alpha \leq 1\}$ be a family of closed intervals. Suppose that $A_1 \neq 0$ and $\{A_\alpha\}$ is decreasing with respect to α. Then $\{A_\alpha\}$ can induce a fuzzy number $\tilde{a}$.

Proposition 2.12:
(Wu Hsein-Chung, 2000). Let $\{A_\alpha = [l_\alpha, u_\alpha] : 0 \le \alpha \le 1\}$ be a family of closed intervals. Suppose that the following conditions are satisfied:

(i) $A_1 \neq 0$,

(ii) $\{A_\alpha\}$ is decreasing with respect to α,

(iii) l_α and u_α are left-continuous with respect to α.

Then $\{A_\alpha\}$ can induce a closed fuzzy number $\tilde{a}$ and $\tilde{a}_\alpha = A_\alpha$.

Proposition 2.13:
(Wu Hsein-Chung, 2000). (i) Let $A_\alpha = \{x : \mu(x) \ge \alpha\}$. Then $\cap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha$ for $\alpha_n \uparrow \alpha$.

(ii) If $\tilde{a}$ is a closed fuzzy number then $\tilde{a}^L_{\alpha_n} \uparrow \tilde{a}^L_\alpha$ and $\tilde{a}^U_{\alpha_n} \downarrow \tilde{a}^U_\alpha$ for $\alpha_n \uparrow \alpha$ (i.e. left-continuous with respect to α).

Proposition 2.14:
(Rudin, W., 1986) (Lebesgue's Theorem). Let f be a bounded function on $[a,b]\times[c,d]$. Then, f is Riemann-integrable on $[a,b]\times[c,d]$, if and only if f is continuous a.e. on $[a,b]\times[c,d]$.

Proposition 2.15:
(Rudin, W., 1986). Let f be a bounded function defined on $[a,b]\times[c,d]$. If f is Riemann-integrable on $[a,b]\times[c,d]$ then the Lebesgue integral and Riemann integral are identical.

Proposition 2.16:

Let $\{f_n(x,y)\}$ be a sequence of bounded function. Suppose that $\lim_{n\to\infty} f_n(x,y) = f(x,y), \lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b, \lim_{n\to\infty} c_n = c, \lim_{n\to\infty} d_n = d$ and $f_n(x,y)$ is Riemann -integrable on $[a_n,b_n]\times[c_n,d_n]$ for all n.

(i) If $0 \le f_1 \le \ldots \le f_n \le f_{n+1} \le \ldots, a_n \le b_n, b_n \le b_{n+1}$, $c_n \le d_n$, $d_n \le d_{n+1}$ and $a_n \ge a_{n+1}$, $c_n \ge c_{n+1}$ then

$$\lim_{n\to\infty} \int_{a_n}^{b_n} \int_{c_n}^{d_n} f_n(x,y)dxdy = \int_a^b \int_c^d f(x,y)dxdy. \tag{14}$$

(ii) If $f_1 \ge \ldots \ge f_n \ge f_{n+1} \ge \ldots \ge 0, a_n \le b_n, b_n \ge b_{n+1}$, $c_n \le d_n$, $d_n \le d_{n+1}$ and $a_n \le a_{n+1}$, $c_n \le c_{n+1}$ then

$$\lim_{n\to\infty} \int_{a_n}^{b_n} \int_{c_n}^{d_n} f_n(x,y)dxdy = \int_a^b \int_c^d f(x,y)dxdy. \tag{15}$$

(iii) If $f_1 \le \ldots \le f_n \le f_{n+1} \le \ldots \le 0, a_n \le b_n, b_n \ge b_{n+1}$, $c_n \le d_n$, $d_n \ge d_{n+1}$ and $a_n \le a_{n+1}$, $c_n \le c_{n+1}$ then

$$\lim_{n\to\infty} \int_{a_n}^{b_n} \int_{c_n}^{d_n} f_n(x,y)dxdy = \int_a^b \int_c^d f(x,y)dxdy. \tag{16}$$

(iv) If $0 \ge f_1 \ge \ldots \ge f_n \ge f_{n+1} \ge \ldots, a_n \le b_n, b_n \le b_{n+1}$, $c_n \le d_n$, $d_n \le d_{n+1}$ and $a_n \ge a_{n+1}$, $c_n \ge c_{n+1}$ then

$$\lim_{n\to\infty}\int_{a_n}^{b_n}\int_{c_n}^{c_n} f_n(x,y)dxdy = \int_a^b\int_c^d f(x,y)dxdy. \tag{17}$$

Proof:
(i) From proposition 2.14, we can use the Lebesgue's monotone convergence theorem (Royden, 1968). Since

$$\int_{a_n}^{b_n}\int_{c_n}^{d_n} f_n(x,y)dxdy = \int_{a_n}^{t}\int_{c_n}^{k} f_n(x,y)dxdy + \int_{a_n}^{t}\int_{k}^{d_n} f_n(x,y)dxdy + \int_{t}^{b_n}\int_{c_n}^{k} f_n(x,y)dxdy + \int_{t}^{b_n}\int_{k}^{d_n} f_n(x,y)dxdy$$

we consider

$$\int_{a_n}^{t}\int_{c_n}^{k} f_n(x,y)dxdy = \iint_R 1_{[a_n,t]\times[c_n,k]}(x,y).f_n(x,y)dxdy \equiv \iint_R g_n(x,y)dxdy \ .$$

Then $0 \le g_n(x,y) \le g_{n+1}(x,y)$ (since $f_n(x,y) \le f_{n+1}(x,y)$, $a_n \ge a_{n+1}$, $c_n \ge c_{n+1}$ and

$lim_{n\to\infty} g_n(x,y) = 1_{[a,t]\times[c,k]}(x,y).f(x,y))$.

By the Lebesgue's monotone convergence theorem, we have

$$lim_{n\to\infty}\int_{a_n}^{t}\int_{c_n}^{k} f_n(x,y)dxdy = lim_{n\to\infty}\iint_R g_n(x,y)dxdy = \iint_R g(x,y)dxdy =$$

$$\iint_R 1_{[a,t]\times[c,k]}(x,y).f(x,y)dxdy = \int_a^t\int_c^k f(x,y)dxdy$$

Similarly, we have

$$lim_{n\to\infty}\int_{a_n}^{t}\int_{k}^{d_n} f_n(x,y)dxdy = \int_a^t\int_k^d f(x,y)dxdy \ ,$$

$$lim_{n\to\infty}\int_{t}^{b_n}\int_{c_n}^{k} f_n(x,y)dxdy = \int_t^b\int_c^k f(x,y)dxdy \ ,$$

$$lim_{n\to\infty}\int_{t}^{b_n}\int_{k}^{d_n} f_n(x,y)dxdy = \int_t^b\int_k^d f(x,y)dxdy \ .$$

The result follows immediately.

(ii) Let $g_n(x,y) \equiv 1_{[a_n,t]\times[c_n,k]}(x,y).f_n(x,y)$. Then $g_1 \ge ... \ge g_n \ge g_{n+1} \ge ... \ge 0$. Thus, $g_n = |g_n| \le g_1$ ($g_1 \ge 0$) and $\iint_R g_1(x,y)dxdy = \int_{a_1}^{t}\int_{c_1}^{k} f_1(x,y)dxdy$ is Lebesgue-integrable. The result follows immediately from the Lebesgue's dominated convergence theorem.

(iii) Let $g_n = -f_n$. Then the result follows from (ii).

(iv) Let $g_n = -f_n$. then the result follows from (i).

3 The Full Fuzzy Double Integrals:

In this section, we develop the concept of a fuzzy Riemann integral of the second kind mentioned in (Wu Hsein-Chung, 2000) to introduce a full fuzzy Riemann double integral (FFDI).

Definition 3.1:
We say that

(i) $\tilde{f}(x,y)$ is a fuzzy function if $\tilde{f}: X \times Y \to F$;

(ii) $\tilde{f}(x,y)$ is a closed-fuzzy function if $\tilde{f}: X \times Y \to F_{cl}$;

(iii) $\tilde{f}(x,y)$ is a bounded-fuzzy function if $\tilde{f}: X \times Y \to F_b$.

Definition 3.2:

Let $\tilde{f}(x,y)$ be a bounded- and closed-fuzzy-valued function defined on the closed fuzzy real number system and $\tilde{f}(x,y)$ be induced by $\tilde{f}(\tilde{x},\tilde{y})$. Suppose that $\tilde{b} \ge \tilde{a}$.

(i) If $\tilde{f}(x,y)$ is nonnegative and $\tilde{f}_\alpha^L(x,y)$ and $\tilde{f}_\alpha^U(x,y)$ are Riemann-integrable on $[\tilde{a}_\alpha^U,\tilde{b}_\alpha^L]$ and $[\tilde{a}_\alpha^L,\tilde{b}_\alpha^U]$, respectively, for all α then we let

$$A_\alpha = \begin{cases} [\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^L(x,y)dxdy, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy] & if \quad \tilde{b}_\alpha^L > \tilde{a}_\alpha^U, \tilde{d}_\alpha^L > \tilde{c}_\alpha^U \\ [0, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy] & if \quad \tilde{b}_\alpha^L \le \tilde{a}_\alpha^U, \tilde{d}_\alpha^L \le \tilde{c}_\alpha^U. \end{cases} \tag{18}$$

(ii) If $\tilde{f}(x,y)$ is nonpositive and $\tilde{f}_\alpha^L(x,y)$ and $\tilde{f}_\alpha^U(x,y)$ are Riemann-integral on $[\tilde{a}_\alpha^L,\tilde{b}_\alpha^U]$ and $[\tilde{a}_\alpha^U,\tilde{b}_\alpha^L]$ respectively, for all α then we let

$$A_\alpha = \begin{cases} [\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^L(x,y)dxdy, \int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^U(x,y)dxdy] & if \quad \tilde{b}_\alpha^L > \tilde{a}_\alpha^U, \tilde{d}_\alpha^L > \tilde{c}_\alpha^U \\ [\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^L(x,y)dxdy, 0] & if \quad \tilde{b}_\alpha^L \le \tilde{a}_\alpha^U, \tilde{d}_\alpha^L \le \tilde{c}_\alpha^U. \end{cases} \tag{19}$$

Under the above conditions, we say that $\tilde{f}(\tilde{x},\tilde{y})$ is full fuzzy Riemann-integrable on the fuzzy interval $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$, and the membership function of $\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$ is defined by,

$$\mu_{\tilde{I}}(r) = \sup_{0\le\alpha\le1} \alpha 1_{A_\alpha}(r), \tag{20}$$

for $r \in A_0$.

Theorem 3.1:

If each of $\tilde{f}(\tilde{x},\tilde{y})$ and $\tilde{g}(\tilde{x},\tilde{y})$ is closed and bounded-fuzzy-valued and $\tilde{f}(x,y), \tilde{g}(x,y)$ are full fuzzy double integrable on $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$ then $\tilde{f}(x,y) \oplus \tilde{g}(x,y)$ is a full fuzzy double integrable on $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$. Moreover, we have

$$\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} (\tilde{f}(\tilde{x},\tilde{y}) \oplus \tilde{g}(\tilde{x},\tilde{y}))d\tilde{x}d\tilde{y} = \int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} \oplus \int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{g}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$$

Proof:

Without loss of generality we suppose that $\tilde{f}$ and $\tilde{g}$ are nonnegative fuzzy functions. Let $\tilde{h}(\tilde{x},\tilde{y}) = \tilde{f}(\tilde{x},\tilde{y}) \oplus \tilde{g}(\tilde{x},\tilde{y})$. Then $\tilde{h}(\tilde{x},\tilde{y})$ is a closed-fuzzy-valued function and by definition 3.3 if $\tilde{b}_\alpha^L > \tilde{a}_\alpha^U$ and $\tilde{d}_\alpha^L > \tilde{c}_\alpha^U$ *we have* $(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} \oplus \int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{g}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y})_\alpha =$

$$[\int_{\tilde{a}_\alpha{}^U}^{\tilde{b}^L}\int_{\tilde{c}^U}^{\tilde{d}^L} \tilde{f}^L(x,y)dxdy + \int_{\tilde{a}_\alpha{}^U}^{\tilde{b}_\alpha{}^L}\int_{\tilde{c}_\alpha{}^U}^{\tilde{d}_\alpha{}^L} \tilde{g}^L(x,y)dxdy, \int_{\tilde{a}_\alpha{}^L}^{\tilde{b}_\alpha{}^U}\int_{\tilde{c}_\alpha{}^L}^{\tilde{d}_\alpha{}^U} \tilde{f}^U(x,y)dxdy + \int_{\tilde{a}_\alpha{}^L}^{\tilde{b}_\alpha{}^U}\int_{\tilde{c}_\alpha{}^L}^{\tilde{d}_\alpha{}^U} \tilde{g}^U(x,y)dxdy]$$

$$=[\int_{\tilde{a}_\alpha{}^U}^{\tilde{b}_\alpha{}^L}\int_{\tilde{c}_\alpha{}^U}^{\tilde{d}_\alpha{}^L} (\tilde{f}_\alpha{}^L(x,y) + \tilde{g}_\alpha{}^L(x,y))dxdy, \int_{\tilde{a}_\alpha{}^L}^{\tilde{b}_\alpha{}^U}\int_{\tilde{c}_\alpha{}^L}^{\tilde{d}_\alpha{}^U} (\tilde{f}_\alpha{}^U(x,y) + \tilde{g}_\alpha{}^U(x,y))dxdy]$$

$$=[\int_{\tilde{a}_\alpha{}^U}^{\tilde{b}_\alpha{}^L}\int_{\tilde{c}_\alpha{}^U}^{\tilde{d}_\alpha{}^L} (\tilde{f}\ \ (x,y) \oplus \tilde{g}\ \ (x,y))^L dxdy, \int_{\tilde{a}_\alpha{}^L}^{\tilde{b}_\alpha{}^U}\int_{\tilde{c}^L}^{\tilde{d}^U} (\tilde{f}\ \ (x,y) + \tilde{g}\ \ (x,y))_\alpha{}^U dxdy]$$

$$=(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} (\tilde{f}(\tilde{x},\tilde{y}) \oplus \tilde{g}(\tilde{x},\tilde{y}))d\tilde{x}d\tilde{y})_\alpha$$

The proof is complete from definition 2.7(*iii*). The other cases can be proved similarly.

We consider two given real numbers *x* and *y*. We can induce fuzzy numbers $\tilde{x}$ and $\tilde{y}$ with membership functions $\mu_{\tilde{x}}(r)$ and $\mu_{\tilde{y}}(r)$ such that $\mu_{\tilde{x}}(x) = 1$ and $\mu_{\tilde{x}}(r) < 1$ for $r \neq x$ (i.e. $r = x$ the unique global maximum of the membership function). We call $\tilde{x}$ and $\tilde{y}$ as fuzzy real numbers induced by the real numbers *x* and *y* respectively.

Let $\tilde{f}$ be a double fuzzy function. Then we can induce a new fuzzy-valued function $\tilde{f}: R \times R \to F \times F$ by $\tilde{f}(x,y) = \tilde{f}(\tilde{x},\tilde{y})$. Now, suppose that $\tilde{b} \succeq \tilde{a}$ and $\tilde{c} \succeq \tilde{d}$, we shall discuss the fuzzy Riemann integrals on the fuzzy interval formed by $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, $\tilde{d}$ (denoted as $[\tilde{a},\tilde{b}]$ and $[\tilde{c},\tilde{d}]$), where $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, $\tilde{d}$ are four closed fuzzy real numbers induced by four real numbers *a, b, c* and *d* respectively.

In order to define the full fuzzy Riemann integral, we need to consider the "length" between $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$, $\tilde{d}$ for $\tilde{b} \succ \tilde{a}$ and $\tilde{c} \succ \tilde{d}$ ($\tilde{b} \succ \tilde{a}$ means $\tilde{b} \succeq \tilde{a}$ and $\tilde{b}_\alpha^L \geq \tilde{a}_\alpha^U$ for all α). Now $(\tilde{b}-\tilde{a})_\alpha^L = \tilde{b}_\alpha^L - \tilde{a}_\alpha^U$ and $(\tilde{b}-\tilde{a})_\alpha^U = \tilde{b}_\alpha^U - \tilde{a}_\alpha^L$ and $(\tilde{d}-\tilde{c})_\alpha^L = \tilde{d}_\alpha^L - \tilde{c}_\alpha^U$ and $(\tilde{d}-\tilde{c})_\alpha^U = \tilde{d}_\alpha^U - \tilde{c}_\alpha^L$. We shall consider the interval $[\tilde{a}_\alpha^U, \tilde{b}_\alpha^L]$ and $[\tilde{c}_\alpha^U, \tilde{d}_\alpha^L]$ for the lower bound case and the intervals $[\tilde{a}_\alpha^L, \tilde{b}_\alpha^U]$ and $[\tilde{c}_\alpha^L, \tilde{d}_\alpha^U]$ for the upper bound case.

Lemma 3.1:

(i) Let $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers. Suppose that $\tilde{b} \succeq \tilde{a}$. Then we only have two cases.

(a) $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^L$ for all $\alpha \in [0,1]$.

(b) $\exists \alpha_0 \in [0,1]$ such that $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^L$ for $0 \leq \alpha \leq \alpha_0$ and $\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^L$ for $1 \geq \alpha > \alpha_0$.

(ii) Suppose that $\alpha_n \uparrow \alpha$ and $\infty > \alpha > \alpha_0$. Then $\exists N$ such that $n > N$ implies $\alpha_n > \alpha_0$.

Proof:
(i) This result is obvious, since $\tilde{a}_\alpha^U$ is decreasing with respect to α and $\tilde{b}_\alpha^L$ is increasing with respect to α.

(ii) Since $\alpha_n \uparrow \alpha$, we have $\sup \alpha_n = \alpha$. If $\alpha_n \le \alpha_0$ for all n then $\alpha = \sup \alpha_n \le \alpha_0$ which contradicts $\alpha > \alpha_0$.

Theorem 3.2:
(Wu Hsein-Chung, 2000). Let $\tilde{f}(\tilde{x},\tilde{y})$ be a bounded- and closed-fuzzy-valued function and $\tilde{f}(\tilde{x},\tilde{y})$ be induced by $\tilde{f}(x,y)$. Suppose that $\tilde{f}(x,y)$ is nonnegative or nonpositive and $(\tilde{x},\tilde{y}) \in (FFDI)$ on $[\tilde{a},\tilde{b}] \times [\tilde{c},\tilde{d}]$.

(i) If $\tilde{f}(x,y)$ is nonnegative then $\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$ is a closed fuzzy number and

$$(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y})_\alpha = \begin{cases} [\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^L(x,y)dxdy, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy] & if \tilde{b}_\alpha^L > \tilde{a}_\alpha^U, \tilde{c}_\alpha^L > \tilde{d}_\alpha^U \\ [0, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxd] & if \quad \tilde{b}_\alpha^L \le \tilde{a}_\alpha^U, \tilde{d}_\alpha^L \le \tilde{c}_\alpha^U. \end{cases} \tag{21}$$

(ii) If $\tilde{f}(x,y)$ is nonpositive then $\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$ is a closed fuzzy number and

$$(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y})_\alpha = \begin{cases} [\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^L(x,y)dxdy, \int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^U(x,y)dxdy] & if \tilde{b}_\alpha^L > \tilde{a}_\alpha^U, \tilde{d}_\alpha^L > \tilde{c}_\alpha^U \\ \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^L(x,y)dxdy, 0] & if \quad \tilde{b}_\alpha^L \le \tilde{a}_\alpha^U, \tilde{d}_\alpha^L \le \tilde{c}_\alpha^U. \end{cases} \tag{22}$$

Proof:.
(i) Let

$$A_\alpha = \begin{cases} [\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^L(x,y)dxdy, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy] & if \quad \tilde{b}_\alpha^L > \tilde{a}_\alpha^U, \tilde{c}_\alpha^L > \tilde{d}_\alpha^U \\ [0, \int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy] & if \quad \tilde{b}_\alpha^L \le \tilde{a}_\alpha^U \tilde{d}_\alpha^L \le \tilde{c}_\alpha^U \end{cases} \tag{23}$$

First of all, we prove that $\{A_\alpha\}$ is decreasing with respect to α. Without loss of generality, we consider case (b) in lemma 3.1(*i*). Then $[\tilde{a}_\alpha^U, \tilde{b}_\alpha^L]$ and $[\tilde{c}_\alpha^U, \tilde{d}_\alpha^L]$ are increasing intervals with respect to α for $\alpha > \alpha_0$. Since $\tilde{f}_\alpha^L(x,y)$ is nonnegative and increasing with respect to α, $\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L} \tilde{f}_\alpha^L(x,y)dxdy$ is also increasing with respect to α for $\alpha > \alpha_0$. On the other hand, the left end point of the closed interval *A* is equal to zero for $\alpha \le \alpha_0$ since $\tilde{a}_\alpha^U \ge \tilde{b}_\alpha^L$ and $\tilde{c}_\alpha^U \ge \tilde{d}_\alpha^L$ for $\alpha \le \alpha_0$. Similarly, $\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U} \tilde{f}_\alpha^U(x,y)dxdy$ is decreasing with respect to α, since $\tilde{f}^U_\alpha$ and the intervals $[\tilde{a}^L_\alpha, \tilde{b}^U_\alpha]$ and $[\tilde{c}^L_\alpha, \tilde{d}^U_\alpha]$ are decreasing with respect to α. Thus, $\{A_\alpha\}$ is decreasing with respect to α. Now for $\alpha_n \uparrow \alpha$,

(a) if $\alpha \le \alpha_0$ then $\alpha_n \le \alpha \le \alpha_0$ and the left end point of the closed interval A_α is equal to zero by lemma 3.1(i)(b);

(b) if $\alpha > \alpha_0$ then, by lemma 3.1(ii), $\exists N$ such that $\alpha_n > \alpha_0$ for $n > N$. Thus, from proposition 2.15(i) and lemma 3.1(i)(b), we have

$$\lim_{n\to\infty} \int_{\tilde{a}^U_{\alpha_n}}^{\tilde{b}^L_{\alpha_n}} \int_{\tilde{c}^U_{\alpha_n}}^{\tilde{d}^L_{\alpha_n}} \tilde{f}^U_{\alpha_n}(x,y)dxdy = \int_{\tilde{a}^U_\alpha}^{\tilde{b}^L_\alpha} \int_{\tilde{c}^U_\alpha}^{\tilde{d}^L_\alpha} \tilde{f}^U_\alpha(x,y)dxdy$$

Similarly, from proposition 2.15(ii), we have

$$\lim_{n\to\infty} \int_{\tilde{a}^L_{\alpha_n}}^{\tilde{b}^U_{\alpha_n}} \int_{\tilde{c}^L_{\alpha_n}}^{\tilde{d}^U_{\alpha_n}} \tilde{f}^U_{\alpha_n}(x,y)dxdy = \int_{\tilde{a}^L_\alpha}^{\tilde{b}^U_\alpha} \int_{\tilde{c}^L_\alpha}^{\tilde{d}^U_\alpha} \tilde{f}^U_\alpha(x,y)dxdy$$

Then the result follows from proposition 2.11.

(ii) By proposition 2.15 (iii) and (iv), the result follows from the same techniques as (i).

Definition 3.3:

Let $\tilde{f}(\tilde{x},\tilde{y})$ be a bounded-and closed-fuzzy-valued function and $\tilde{f}(x,y)$ be induced by $\tilde{f}(\tilde{x},\tilde{y})$. Then $\tilde{f}^+(\tilde{x},\tilde{y})$ is a nonnegative bounded-and closed-fuzzy-valued function and $\tilde{f}^-(\tilde{x},\tilde{y})$ is a nonpositive bounded-and closed-fuzzy-valued function by Remark 2.2.

If $\tilde{f}^+(\tilde{x},\tilde{y})$ and $\tilde{f}^-(\tilde{x},\tilde{y})$ are full fuzzy double integrable on $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$ then we say that $\tilde{f}(\tilde{x},\tilde{y})$ is fuzzy Riemann-integrable on the fuzzy interval $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$, and $\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$ is defined by

$$\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} = \int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}^+(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y} \oplus \int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}^-(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$$

Remark 3.1:
(i) By proposition 2.3, remark 2.2 and theorem 3.2, $\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}$ is a closed fuzzy number and its α-level set is

$$\left(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}} \tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y}\right)_\alpha = \begin{cases} \left[\int_{\tilde{a}^U_\alpha}^{\tilde{b}^L_\alpha}\int_{\tilde{c}^U_\alpha}^{\tilde{d}^L_\alpha} (\tilde{f}^+(\tilde{x},\tilde{y}))^L_\alpha d\tilde{x}d\tilde{y} + \int_{\tilde{a}^L_\alpha}^{\tilde{b}^U_\alpha}\int_{\tilde{c}^L_\alpha}^{\tilde{d}^U_\alpha} (\tilde{f}^-(\tilde{x},\tilde{y}))^L_\alpha d\tilde{x}d\tilde{y},\right. & \\ \left.\int_{\tilde{a}^L_\alpha}^{\tilde{b}^U_\alpha}\int_{\tilde{c}^L_\alpha}^{\tilde{d}^U_\alpha} (\tilde{f}^+(\tilde{x},\tilde{y}))^U_\alpha d\tilde{x}d\tilde{y} + \int_{\tilde{a}^U_\alpha}^{\tilde{b}^L_\alpha}\int_{\tilde{c}^U_\alpha}^{\tilde{d}^L_\alpha} (\tilde{f}^+(\tilde{x},\tilde{y}))^U_\alpha d\tilde{x}d\tilde{y}\right] & if \tilde{b}^L_\alpha > \tilde{a}^U_\alpha, \tilde{d}^L_\alpha > \tilde{c}^U_\alpha \\ \left[\int_{\tilde{a}^L_\alpha}^{\tilde{b}^U_\alpha}\int_{\tilde{c}^L_\alpha}^{\tilde{d}^U_\alpha} (\tilde{f}^-(\tilde{x},\tilde{y}))^L_\alpha d\tilde{x}d\tilde{y}, \int_{\tilde{a}^L_\alpha}^{\tilde{b}^U_\alpha}\int_{\tilde{c}^L_\alpha}^{\tilde{d}^U_\alpha} (\tilde{f}^+(\tilde{x},\tilde{y}))^U_\alpha d\tilde{x}d\tilde{y}\right] & if \tilde{b}^L_\alpha \le \tilde{a}^U_\alpha, \tilde{d}^L_\alpha \le \tilde{c}^U_\alpha. \end{cases} \quad (24)$$

4 Computational Methods and Examples:

Let $\tilde{f}(\tilde{x},\tilde{y})$ be a closed-and bounded-fuzzy-valued function defined on the closed fuzzy real number system and $\tilde{f}(\tilde{x},\tilde{y})$ be induced by $\tilde{f}(x,y)$. Suppose that $\tilde{f}(\tilde{x},\tilde{y})$ is nonnegative and $\tilde{f}(\tilde{x},\tilde{y})$ is full fuzzy double integrable on $[\tilde{a},\tilde{b}]\times[\tilde{c},\tilde{d}]$. From (21) in theorem 3.2 we have

$$(\int_{\tilde{a}}^{\tilde{b}}\int_{\tilde{c}}^{\tilde{d}}\tilde{f}(\tilde{x},\tilde{y})d\tilde{x}d\tilde{y})_\alpha=[\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L}\tilde{f}_\alpha^L(x,y)dxdy,\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U}\tilde{f}_\alpha^U(x,y)dxdy].$$

Let

$$\int_{\tilde{a}_\alpha^U}^{\tilde{b}_\alpha^L}\int_{\tilde{c}_\alpha^U}^{\tilde{d}_\alpha^L}\tilde{f}_\alpha^L(x,y)dxdy\equiv g(\alpha), \tag{25}$$

and

$$\int_{\tilde{a}_\alpha^L}^{\tilde{b}_\alpha^U}\int_{\tilde{c}_\alpha^L}^{\tilde{d}_\alpha^U}\tilde{f}_\alpha^U(x,y)dxdy\equiv h(\alpha). \tag{26}$$

Hence $\tilde{I}_\alpha=[g(\alpha),h(\alpha)]$. By using proposition 2.4 and 2.6 in section 2, the membership function of $\tilde{I}$ is as follows:

$$\mu(r)=\sup_{0\le\alpha\le1}\alpha 1_{I_\alpha}(r)=\max_{0\le\alpha\le1}\alpha 1_{I_\alpha}(r)=\max\{\alpha\,|\,0\le\alpha\le1, r\in I_\alpha\}. \tag{27}$$

Therefore

$$\mu(r)=\max\alpha \tag{28}$$

$$St.\begin{cases}\alpha\le1\\ g(\alpha)\le r\\ h(\alpha)\ge r\\ \alpha\ge0\end{cases}$$

Now, if *g(a)* and *h(a)* are computable analytically, since *g(a)* and *h(a)* be increasing and decreasing functions respectively with α, then:

(i) if $g(1)\le r\le h(1)$, then $\mu(r)=1$,

(ii) if $r<g(1)$ then the constraint $h(\alpha)\ge r$ is redundant, since *h(a)* is decreasing.

That is,

$h(\alpha)\ge h(1)\ge g(1)>r$ for $\alpha\in[0,1]$. Thus, we solve the following relaxed nonlinear program

$$\mu(r) = \max \alpha \tag{29}$$
$$St.\begin{cases} \alpha \le 1 \\ g(\alpha) \le r \\ \alpha \ge 0 \end{cases}$$

Therefore,

$\mu(r) = \max\{\alpha \mid 0 \le \alpha \le 1$, α is a root of $g(\alpha) = r\} \simeq \max\{\alpha \mid 0 \le \alpha \le 1$, $|\underline{I}_{\alpha}^{m,n} - r| \le \varepsilon\}$,

(iii) if $r > h(1)$ then the constraint $g(\alpha) \le r$ is redundant, since *g(a)* is decreasing.

That is,

$g(\alpha) \le g(1) \le h(1) < r$ for $\alpha \in [0,1]$. Thus, we solve the relaxed nonlinear program as follows

$$\mu(r) = \max \alpha \tag{30}$$
$$St.\begin{cases} \alpha \le 1 \\ h(\alpha) \ge r \\ \alpha \ge 0 \end{cases}$$

Therefore,

$\mu(r) = \max\{\alpha \mid 0 \le \alpha \le 1$, α is a root of $h(\alpha) = r\}$ $\simeq \max\{\alpha \mid 0 \le \alpha \le 1$, $|\overline{I}_{\alpha}^{m,n} - r| \le \varepsilon\}$,

where $\underline{I}_{\alpha}^{m,n}$ and $\overline{I}_{\alpha}^{m,n}$ are the approximations of *g(a)* and *h(a)* respectively which are computed by the double Simpson's rule and ε is a given positive value.

Theorem 4.1:

(Burden, 2001). Let $x_i = a + ih$ and $y_i = c + jk$, for each $i = 0,1,...,n$ and $j = 0,1,...,m$ where $h = (b-a)/n$ and $k = (d-c)/m$. If $\partial^4 f/\partial x^4$ and $\partial^4 f/\partial y^4$ are continuous, for some $(\overline{\eta},\overline{\mu})$ and $(\hat{\eta},\hat{\mu})$ in $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, then the error term E of the composite double Simpson's rule has the form:

$$E = -\frac{(d-c)(b-a)}{180}[h^4 \frac{\partial^4 f}{\partial x^4}(\overline{\eta},\overline{\mu}) + k^4 \frac{\partial^4 f}{\partial y^4}(\hat{\eta},\hat{\mu})]. \tag{31}$$

Let $\underline{E}_{\alpha}^{m,n} = \underline{I}_{\alpha}^{m,n} - \underline{I}_{\alpha}$ and $\overline{E}_{\alpha}^{m,n} = \overline{I}_{\alpha}^{m,n} - \overline{I}_{\alpha}$ then by relation (31) we have,

$|\underline{E}_{\alpha}^{m,n}| \le \underline{U}_{\alpha}^{m,n}$ and $|\overline{E}_{\alpha}^{m,n}| \le \overline{U}_{\alpha}^{m,n}$, where,

$$\underline{U}_{\alpha}^{m,n} = \frac{(\tilde{d}_{\alpha}^{L} - \tilde{c}_{\alpha}^{U})(\tilde{b}_{\alpha}^{L} - \tilde{a}_{\alpha}^{U})}{180}[h^4 M_{1_{\alpha}}^{m,n} + k^4 M_{2_{\alpha}}^{m,n}], \tag{32}$$

$$\overline{U}_{\alpha}^{m,n} = \frac{(\tilde{d}_{\alpha}^{U} - \tilde{c}_{\alpha}^{L})(\tilde{b}_{\alpha}^{U} - \tilde{a}_{\alpha}^{L})}{180}[h^4 N_{1_{\alpha}}^{m,n} + k^4 N_{2_{\alpha}}^{m,n}], \tag{33}$$

and $|\frac{\partial^4 \tilde{f}_{\alpha}^{L}(x,y)}{\partial x^4}| \leq M_{1_{\alpha}}^{m,n}$, $|\frac{\partial^4 \tilde{f}_{\alpha}^{L}(x,y)}{\partial y^4}| \leq M_{2_{\alpha}}^{m,n}$ and $|\frac{\partial^4 \tilde{f}_{\alpha}^{U}(x,y)}{\partial x^4}| \leq N_{1_{\alpha}}^{m,n}$, $|\frac{\partial^4 \tilde{f}_{\alpha}^{U}(x,y)}{\partial y^4}| \leq N_{2_{\alpha}}^{m,n}$.

By applying the relations (32) and (33) the following algorithm is introduced. In this algorithm ε and *d* are positive given values.

Algorithm:

1) Read *m, n, ε, d, r*

2) For α =1:- *d*:0 do the following steps:

2.1) Compute $\tilde{a}_{\alpha}^{L}$, $\tilde{a}_{\alpha}^{U}$, $\tilde{b}_{\alpha}^{L}$, $\tilde{b}_{\alpha}^{L}$, $\tilde{c}_{\alpha}^{L}$, $\tilde{c}_{\alpha}^{U}$, $\tilde{d}_{\alpha}^{L}$, $\tilde{d}_{\alpha}^{U}$, $\underline{U}_{\alpha}^{m,n}$, $\overline{U}_{\alpha}^{m,n}$,

2.2)Calculate $\underline{I}_{\alpha}$ and $\overline{I}_{\alpha}$ by using the double Simpson's rule and call them $\underline{I}_{\alpha}^{m,n}$ and $\overline{I}_{\alpha}^{m,n}$ respectively.

2.3) If $\underline{I}_1 \leq r \leq \overline{I}_1$ then $\mu_{\tilde{I}}(r) = 1$ and go to 3.

if $r < \underline{I}_1$ then if $|\underline{I}_{\alpha}^{m,n} - r| < \varepsilon$ set $\mu_{\tilde{I}\,m,n}(r) = \alpha$ and go to (3) If $r > \overline{I}_1$ then if $|\overline{I}_{\alpha}^{m,n} - r| < \varepsilon$ set $\mu_{\tilde{I}\,m,n}(r) = \alpha$ and go to (3).

3) print *m, n,* $\mu_{\tilde{I}\,m,n}(r)$, $\underline{I}_{\alpha}^{m,n}$, $\overline{I}_{\alpha}^{m,n}$, $\underline{U}_{\alpha}^{m,n}$, $\overline{U}_{\alpha}^{m,n}$ and stop.

4.1 Numerical Examples:

In this part, we evaluate two full fuzzy double integrals. The programs have been provided by MATLAB according to the mentioned algorithm.

Example 4.1:

Consider fuzzy double integral $\int_{\tilde{1}.4}^{\tilde{2}.0}\int_{\tilde{1}.0}^{\tilde{1}.5} \ln(\tilde{x}+\tilde{y})d\tilde{x}d\tilde{y}$. Let $\tilde{2}.0$, $\tilde{1}.4$, $\tilde{1}.5$ and $\tilde{1}.0$ be triangular fuzzy numbers. In this case, $x = (x-1, x, x+1)$ and $y = (y+1, y, y-1)$, $(\tilde{x})_{\alpha} = [\alpha + x - 1, -\alpha + x + 1]$ and $(\tilde{y})_{\alpha} = [\alpha + y - 1, -\alpha + y + 1]$, $\underline{I}^1 = \overline{I}^1 = \int_{1.4}^{2.0}\int_{1.0}^{1.5} \ln(x+2y)dxdy = 0.323659$ and $\tilde{f}_{\alpha}^{L}(x,y) = ln[(\alpha + x - 1) + (\alpha + y - 1)]$ and $\tilde{f}_{\alpha}^{U}(x,y) = ln[(-\alpha + x + 1) + (-\alpha + y + 1)]$. Let $d = 0.000001$ and $\varepsilon = 0.0001$.

Table 1 shows the results of the algorithm for r=0.32 , we observe that, $\mu_{\tilde{I}}(r) \cong \mu_{\tilde{I}m,n}(r) = 0.955190$ and $\underline{I}_{\alpha} \cong \underline{I}_{\alpha}^{m,n}(r) = 0.3180039$, $\overline{I}_{\alpha} = \overline{I}_{\alpha}^{m,n} = 0.320202$ with m=12 and n=14.

Table 1: The results of example 4.1 for *r* = 0.32.

m	n	$\mu_{\tilde{I}m,n}(r)$	$\underline{I}_{\alpha}^{m,n}$	$\overline{I}_{\alpha}^{m,n}$	$\underline{U}_{\alpha}^{m,n}$	$\overline{U}\alpha^{m,n}$
2	4	0.944500	0.274501	0.387901	0.0492	0.0642
4	6	0.951450	0.290102	0.340100	0.0336	0.0164
6	8	0.953400	0.299281	0.330099	0.0244	0.0064
8	10	0.954310	0.308311	0.320098	0.0153	0.0036
10	12	0.954840	0.315051	0.320109	0.0086	0.0035
12	14	0.955190	0.3180039	0.320202	0.00057	0.00035

The table 2 shows the results of the algorithm for $r = 0.33$. In this case, we observe that, $\mu_{\tilde{I}}(r) \cong \mu_{\tilde{I}m,n}(r) = 0.955188$ and $\underline{I}_{\alpha} \cong \underline{I}_{\alpha}^{m,n}(r) = 0.320105$, $\overline{I}^{\alpha} = \overline{I}_{\alpha}^{m,n} = 0.293221$ with m=12 and n=14.

Table 2: The results of example 4.1 for r = 0.33.

m	n	$\mu_{\tilde{I}m,n}(r)$	$\underline{I}_{\alpha}^{m,n}$	$\overline{I}_{\alpha}^{m,n}$	$\underline{U}_{\alpha}^{m,n}$	$\bar{U}\alpha^{m,n}$
2	4	0.964500	0.374501	0.287901	0.508	0.358
4	6	0.964535	0.330222	0.290342	0.44	0.56
6	8	0.965340	0.320134	02990515	0.059	0.047
8	10	0.959741	0.320012	0.290021	0.0085	0.0107
10	12	0.959480	0.320103	0.290112	0.0032	0.0045
12	14	0.955188	0.320105	0.293221	0.00032	0.00025

Example 4.2:

Consider full fuzzy double integral $\int_{\tilde{0}}^{\tilde{1}}\int_{\tilde{1}}^{\tilde{2}} \tilde{x}e^{\tilde{y}}d\tilde{x}d\tilde{y}$. Let $\tilde{2}$, $\tilde{1}$ and $\tilde{0}$ be triangular fuzzy numbers. In this case $x = (x-1, x, x+1)$ and $y = (y+1, y, y-1)$, $(\tilde{x})_{\alpha} = [\alpha + x - 1, -\alpha + x + 1]$ and $(\tilde{y})_{\alpha} = [\alpha + y - 1, -\alpha + y + 1]$ $\underline{I}^1 = \overline{I}^1 = \int_0^1\int_1^2 xe^y dxdy = 2.335468$ and $\tilde{f}_{\alpha}^{L}(x,y) = (\alpha + x - 1)e^{(\alpha + y - 1)}$ and

$\tilde{f}_{\alpha}^{U}(x,y) = (-\alpha + x - 1)e^{(-\alpha + y - 1)}$. Let $d = 0.000001$ and $\varepsilon = 0.0001$.

Table 3 shows the results of the algorithm for r = 2.34. We observe that, $\mu_{\tilde{I}}(r) \cong \mu_{\tilde{I}\,m,n}(r) = 0.9955190$ and $\underline{I}_{\alpha} \cong \underline{I}_{\alpha}^{m,n}(r) = 2.335245$, $\overline{I}_{\alpha} \cong \overline{I}_{\alpha}^{m,n}(r) = 2.341460$ with m = 12 and n = 14.

Table 3: The results of example 4.2 for r = 2.34.

m	n	$\mu_{\tilde{I}m,n}(r)$	$\underline{I}_{\alpha}^{m,n}$	$\overline{I}_{\alpha}^{m,n}$	$\underline{U}_{\alpha}^{m,n}$	$\bar{U}\alpha^{m,n}$
2	4	0.9944500	2.341234	2.349878	1.52	1.88
4	6	0.9951450	2.342349	2.349910	0.21	0.40
6	8	0.9953400	2.331212	2.341005	0.061	0.052
8	10	0.9954310	2.330903	2.341156	0.006	0.015
10	12	0.9954840	2.330765	2.341354	0.0030	0.00333
12	14	0.9955190	2.335245	2.341460	0.00025	0.00010

5 Conclusion:

In this work, we proposed a numerical algorithm to evaluate full fuzzy Riemann double integral easily. In this case, we approximated the end points of the α-level set of double fuzzy integral by using the double Simpson's rule and estimated the errors of the results. Also, we computed the value of the membership function in a given point. As an open work, one can develop this idea for evaluating the approximate value of a double fuzzy improper integral with a fuzzy function.

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