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Some Rigidity issues in LP Polygon

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Abstract: The Linear Programming polygon is an undirected and without loop graph denoted by (V, E) with vertices V and edges E and the framework of the LP polygon is $(V, E, \mathbf{p})$. Some combinatorial issues of LP polygon are unfolded by the properties of rigidity matrix of the polygon.

Key words:

INTRODUCTION

Relevance of convex polyhedra to linear programming is obvious. That is, the feasible solution space for a linear programming problem is a polyhedron *P*.

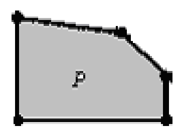

Fig. 1: Polyhedron in 2-Space.

A d -polyhedron is called simple if every vertex of *P* belongs to precisely d edges. Simple polyhedra correspond to non-degenerate linear programming problems. As a convex polyhedron is the intersection P of finite number of closed half spaces in R^d. *P* is a d -dimensional polyhedron if the points of P affinely span R^d, where 2-dimensional polyhedron is polygon generated by a linear programming problem in two variables. A face *F* of a d -polyhedron *P* is the intersection of *P* with a supporting hyperplane. *F* itself is a polyhedron of some lower dimension; like vertex are 0-dimensional polyhedron, edge are 1-dimensional polyhedron and polygons are 2-dimensional polyhedrons.

The set of vertices and edges of P can be regarded as an abstract graph denoted by $G(P)$. We will denote, as by Kalai (1987), $f_k(P)$ the number of k -faces of *P*. The vector ($f_0(P), f_1(P), \ldots, f_d(P)$) is called the f -vector of *P*. Euler's famous formula $V - E + F = 2$ gives a connection between the numbers V, E, F, that is, vertices, edges and 2-faces of every 3–polytope.

The objective function ϕ of linear programming problem attains different values on different vertices of *P* and we can say that every face *F* of *P* is itself a polytope and ϕ attains different values on distinct vertices of *F*. Among the vertices of *F* there is a vertex on which ϕ is maximal and again this vertex is the only vertex in *F* which is a local maximum of ϕ in the face *F*.

Hence moving from face to face and from vertex to vertex in search of the optimal solution has far-reaching applications on the understanding of the combinatorial structure of a simple polytope.

2. Incidence and Framework:

In the figure below, the polygon *P* in discussion is convex and is the intersection of a finite number of halfspaces in the plane; these halfspaces are the constraints of a linear programming problem

$$
\begin{aligned}
2x \quad & \le 1000 \\
2x + 4y & \le 3000 \\
10x + 5y & \le 6000 \\
x \ge 0, \ & y \ge 0
\end{aligned}
$$

with the following polygon

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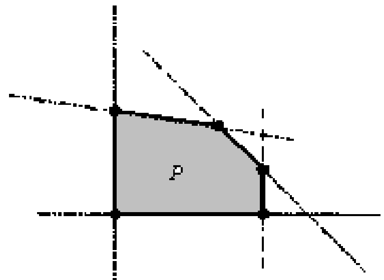

Fig. 2: Linear Programming Polygon.

The set of vertices and the set of edges of P can be regarded as an abstract graph called the graph of P and is denoted by $G(P)$.

3. Rigidity and Framework:

As we have already established that the graph of a polygon under discussion is denoted by $G(P)$, and now we establish that the graph (V,E) consists of the vertex set $V=\{1,2,....,n\}$ and the edge set E which the collection of unordered pair of vertices containing an edge. A framework is a triple (V,E,p) in which (V,E) is the graph and p is listing of point of the space which corresponds to the vertices and is described as $p=\{p_1,p_2,...,p_n\}$. Any deformation of the framework is one parameter family, that is $p=[p_1(t),p_2(t),...,p_n(t)]$ such that the distance from $p_i(t)$ to $p_j(t)$ is kept fixed if both the edges belong to the edge set E, that is to say

$$[p_i(t)-p_j(t)]\bullet[p_i(t)-p_j(t)]=c_{ij} \qquad (1)$$

for all edges of the graph with $p(0)=p$.

Now the framework is said to be rigid whenever $p(t)$ is congruent to p for all t near zero, where $p(t)$ is deformation and p is framework.

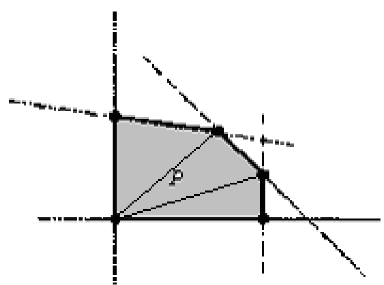

Fig. 3: Triangulated LP Polygon.

$$R(V,E,p)=\begin{pmatrix} [p_1-p_2] & [p_2-p1] & 0 & 0 & 0 \\ [p_1-p_3] & 0 & [p_3-p_1] & 0 & 0 \\ [p_1-p_4] & 0 & 0 & [p_4-p_1] & 0 \\ [p_1-p_5] & 0 & 0 & 0 & [p_5-p_1] \\ 0 & [p_2-p_3] & [p_3-p_2] & 0 & 0 \\ 0 & 0 & [p_3-p_4] & [p_4-p_3] & 0 \\ 0 & 0 & 0 & [p_4-p_5] & [p_5-p_4] \\ 0 & [p_2-p_5] & 0 & 0 & [p_5-p_2] \end{pmatrix}$$

writing in a more compact form by putting $q_{ij}=p_i-p_j$, we have

$$\begin{pmatrix} q_{12} & q_{21} & 0 & 0 & 0 \\ q_{13} & 0 & q_{31} & 0 & 0 \\ q_{14} & 0 & 0 & q_{41} & q_{51} \\ q_{15} & 0 & 0 & 0 & 0 \\ 0 & q_{23} & q_{32} & 0 & 0 \\ 0 & 0 & q_{34} & q_{43} & 0 \\ 0 & 0 & 0 & q_{45} & q_{54} \\ 0 & q_{25} & 0 & 0 & q_{52} \end{pmatrix}$$

$$\begin{pmatrix} -500 & 0 & 500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -500 & -200 & 0 & 0 & 500 & 200 & 0 & 0 & 0 & 0 \\ -310 & -600 & 0 & 0 & 0 & 0 & 310 & 600 & 0 & 0 \\ 0 & -750 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 750 \\ 0 & 0 & 0 & -200 & 0 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 190 & -400 & -190 & 400 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 310 & -150 & -310 & 150 \\ 0 & 0 & 500 & -750 & 0 & 0 & 0 & 0 & -500 & 750 \end{pmatrix}$$

Lemma:

Sum of all vertices index is greater than zero

Theorem:

Dim ker $P(V, E, p) = 3$.

Proof:

Here $P(V, E, p)$ is the rigidity matrix of P, since P is convex and $e = 2n - 3$ where e is the number of edges in the graph and n is the number of vertices in the graph. Since the rigidity matrix $P(V, E, p)$ has $2n$ columns and $e + 1$ rows so the dimension $\dim(P(V, E, p)) = 3$ if and only if $\dim(P(V, E, p))^T = 0$.

Theorem:

A framework of n vertices in d space has $nd - \frac{d(d-1)}{2}$ and implies rigidity.

Proof:

If $P(V, E, p)$ has an infinitesimal flex, then that flex is a solution to the derived equation of (1) above. Since the every framework has $d + 1$ points in general position, the trivial deformations constitute a subspace of $\frac{d(d-1)}{2}$, and so the graph is infinitesimally rigid.

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