

Some Projective Invariants of LP Polytope

Ali Dino Jumani

Department of Mathematics Shah Abdul Latif University Khairpur 66020, Pakistan.

Abstract: Mysteries of Linear Programming are getting wilder and wilder; as the number of variables increases time complexity increases. One obstacle is our inability to "see" in higher dimensional geometry, another striking anonymity is the conjecture posed by W.M. Hirsch in 1957. Different pivot selecting rules has only been the source of improvisation. The projective properties of $S_n(K)$ are those properties of which the expression in every allowable coordinate system \mathfrak{R} is the same, that is to say, the invariance. Theory of harmonic construction and harmonic conjugates and related invariants provides a valuable ability to "see" into LP polytope.

Key words: Convex hull, Harmonic relation, Quadrangle.

INTRODUCTION

A convex polyhedron is the intersection P of a finite number of closed halfspaces in R^d . P is the d -dimensional polyhedron if the points in P affinely span R^d . A convex d -dimensional polytope is a bounded convex d -polyhedron (Kalai, 1997). A nontrivial face F of a d -polyhedron P is the intersection of P with a supporting hyperplane. F itself is a polyhedron of some lower dimension. If the dimension of F is k we call F a k -face of P . The empty set and P itself are regarded as trivial faces. 0 -faces of P are called vertices, 1 -faces are called edges and $(d-1)$ -faces are called facets. Hence a strong relation between convex polytopes and a linear program is very much evident, for more material on convex polytopes.

2. Landmarks:

The classical tools for solving the linear programming problem in practice is the class of simplex algorithm proposed and developed by George Dantzig. The method is based on generating a sequence of bases. The fundamental characteristic of the method is that at some point a basis is reached which provides a solution to the problem. A suitable basis can certify either that the problem has no solution at all or that it is unbounded; otherwise, a basis will be reached which defines optimal solution for both the primal and the dual. Computational Complexity of linear programming had puzzled researchers even before the field of computational complexity started to develop. The question of finding bounds on the diameter and height of polytope is closely related to the complexity of simplex method. Stemming from the study of the diameter of polytope, Kalai developed a randomized simplex algorithm in 1992, and proved the first sub-exponential bound for linear programming and latter improved to $\Delta(d, n) = O(n^{\log d + 2})$ by Kalai and Kleitman. Other sub-exponential randomized algorithms are Clarkson's algorithm which has $O(d^2 \log n)$, Seidel's algorithm leads to the recurrence relation

$$T(d, n) \leq T(d, n-1) + O(d) + \frac{d}{n} (O(dn) + T(d-1, n-1)) \quad (\text{Karmarkar, 1984}),$$

Matousek/Sharir/Welzel algorithm results in sub-exponential bound

$$f(k, n) \leq \min \{ O(k^2 2^k n), e^{\sqrt{k \ln \left(\frac{n}{\sqrt{k}} \right) + O(\sqrt{k} + \ln n)}} \} \quad (\text{Khachiyan, 1979}).$$

Interior-Point Method promoted by N. Karmarkar of Bell Laboratories in 1984 and Ellipsoid Method, the very first polynomial method for linear programming developed by L.G. Khachiyan is also an elegant mathematical construction but all are deficient in practicality.

Corresponding Author: Ali Dino Jumani, Department of Mathematics Shah Abdul Latif University Khairpur 66020, Pakistan.

3. Combinatorics of Linear Programming:

The connection between convex polytope and linear programming is clear. Combinatorial theory of polytopes is the study of their face-structure and in particular their face numbers. The importance of the face number is that the maximum of the objective function ϕ on P is attained at a vertex V and vertices are 0-faces of the polytope. Denoted by $f_i(P)$ the number of i -faces of P . The vector $(f_0(P), f_1(P), \dots, f_d(P))$ is called the f -vector of P . Euler's formula $V - E + F = 2$ is the beginning of a rich theory on face numbers of convex polytopes and related structures. Given a sequence $f = (f_0, f_1, \dots, f_{d-1})$ of nonnegative integers, where $d > 0$ is a fixed integer, put $f_{-1} = 1$ and define $h[f] = (h_0, h_1, \dots, h_d)$ by the relation

$$\sum_{k=0}^d h_k x^{d-k} = \sum_{k=0}^d f_{k-1} (x-1)^{d-k}.$$

If $f = f(K)$ the f -vector of a $(d-1)$ -dimensional simplicial K then $h[f] = h(K)$ is called the h -vector of K . In 1970, McMullen proposed a complete characterization of f -vector of boundary complexes of simplicial polytopes, recently McMullen found an elementary proof which is called g -theorem using polytope algebra.

4. Projective Elements:

Projective geometry includes all propositions of affine geometry that retain their meaning and validity after central projection.

Complete four-point configuration consisting of the four points, six sides and three diagonal points.

Theorem 4.1:

In any complete four-point following are true:

- I. No two of the sides coincide
- II. No four of the sides can be concurrent
- III. No three of the sides can pass through the same diagonal point
- IV. No two of the diagonal points can coincide
- V. No diagonal point can coincide with one of the four given points
- VI. None of the four given points can be collinear with two diagonal points

Theorem 4.2:

At least one harmonic point lies between any two of the four given points.

Proof:

In the algebraic representation, a harmonic tetrad is defined to be a set of four collinear points X, Y, L, M , such that $R(XY, LM) = -1$.

In the algebraic representation,

$$A_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad A_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad A_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad A_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

be the vertices of an arbitrary four-point. Then, the diagonal points $D_2 = \begin{pmatrix} \lambda_0 x_0 + \lambda_2 x_2 \\ \lambda_0 y_0 + \lambda_2 y_2 \\ \lambda_0 z_0 + \lambda_2 z_2 \end{pmatrix}$ and $D_3 = \begin{pmatrix} \lambda_0 x_0 + \lambda_3 x_3 \\ \lambda_0 y_0 + \lambda_3 y_3 \\ \lambda_0 z_0 + \lambda_3 z_3 \end{pmatrix}$ where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are numbers, none of which is zero, such that

$$\begin{cases} \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0 \\ \lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0 \\ \lambda_0 z_0 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 = 0 \end{cases} \quad (1)$$

Adding these vectors, we have

$$D_2 + D_3 = \begin{pmatrix} 2\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \\ 2\lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \\ 2\lambda_0 z_0 + \lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 \end{pmatrix}$$

$$\left\{ D_2 + D_3 = \begin{pmatrix} \lambda_0 x_0 - \lambda_1 x_1 \\ \lambda_0 y_0 - \lambda_1 y_1 \\ \lambda_0 z_0 - \lambda_1 z_1 \end{pmatrix} = \lambda_0 A_0 - \lambda_1 A_1 \right\} \quad (2)$$

Obviously, the point $D_2 + D_3$ is a point on the line $D_2 D_3$. Moreover, from its representation as a linear combination of A_0 and A_1 , it is clear that it is also a point on the line $A_0 A_1$. In other words, it is the intersection of $D_2 D_3$ and $A_0 A_1$, that is, the harmonic point H_1 . Similarly, we have

$$\left\{ D_2 - D_3 = \begin{pmatrix} \lambda_2 x_2 - \lambda_3 x_3 \\ \lambda_2 y_2 - \lambda_3 y_3 \\ \lambda_2 z_2 - \lambda_3 z_3 \end{pmatrix} = \lambda_2 A_2 - \lambda_3 A_3 \right\} \quad (4)$$

which is the coordinate-vector of the intersection of $D_2 D_3$ and $A_2 A_3$, that is, the harmonic point H_1' . From (2) and (3), it is clear that if the line $D_2 D_3$ is parameterized in terms of D_2 and D_3 as base points, then the parameter of H_1 are (1,1) and the parameter of H_1' are (1,-1). Hence

$$R(D_2 D_3, H_1 H_1') = -1.$$

Also, if (XY, LM) is an arbitrary harmonic tetrad, then

$$(XY, LM) \sim (D_2 D_3, H_1 H_1')$$

and therefore

$$R(XY, LM) = R(D_2 D_3, H_1 H_1') = -1.$$

Lemma:

Line at infinity contains at least two diagonal points and two harmonic points.

5. Conclusion:

Projective transformations are to be performed on the data (i.e. convex hull) generated by the C++ implementation of the Double Description Method for generating all vertices and extreme rays of a general convex polyhedron given by a system of linear inequalities $P = \{x \mid Ax \leq b\}$ developed by Professor Komei Fukuda.

REFERENCES

- Coxeter, H.S.M., 1949. The Real Projective Plane, McGraw-Hill Book Company Inc.
- Coxeter, H.S.M., 1948. Regular Polytopes, Methuen & Co. London.
- Dantzig, G.B., 1963. Linear Programming and Extensions, Princeton University Press, Princeton, NJ.
- Fukuda, K., 1995. cdd+ reference manual, Institute of Operations Research, Swiss Federal Institute of Technology, Zurich, Switzerland, , <http://www.ifor.math.ethz.ch/~fukuda/fukuda.html>

- Goldwasser, G., 1995. A Survey of Linear Programming in Randomized Subexponential Time, SIGACT News, 26(2): 96-10.
- Grunbaum, B., 1967. Convex Polytopes, Interscience, London.
- Kalai, G. and D.J. Kleitman, 1992. A quasi-polynomial bound for diameter of graphs of polyhedral, Bulletin of the American Math. Soc., 24: 315-316.
- Kalai, G., 1997. Linear Programming, the simplex algorithm and simple polytopes, Mathematical Programming, 79: 217-233.
- Karmarker, N., 1984. A new polynomial time algorithm for linear programming, Combinatorica, 4: 373-395.
- Khachiyan, L.G., 1979. A polynomial algorithm in linear programming, Doklady Akademii Nauk USSR, 244: 1093-1096.
- Matoušek, J., M. Sharir and E. Welzl, 1992. A subexponential bound for linear programming, In. Proc. 8th Annu. ACM Sympos. Computational Geometry, 1-8.
- McMullen, P., 1996. Weights on Polytopes, Discrete Comput. Geom., 15: 363-388.
- McMullen, P., 1997. *"The number of faces of simplicial Polytopes"*, Israel. J. Math., 9: 559-570.
- McMullen, P., 1993. Hand Book of Convex Geometry, Elsevier Science Publisher B.V. (*personal communication*), 933-988,.
- Stanley, R.P., 1980. *"The Number of Faces of a Simplicial Convex Polytope"*, Advances in Mathematics, 35(3).
- Zeigler, G.M., 1995. Lectures on Polytopes, Graduate Text in Mathematics 152, Springer-Verlag, New York.