Australian Journal of Basic and Applied Sciences, 5(10): 1460-1462, 2011
ISSN 1991-8178

Minkowski Sum of Two LP Polygons and its Relatives

Ali Dino Jumani

Department of Mathematics Shah Abdul Latif University Khairpur 66020, Pakistan.

Abstract: Minkowski sum of two points A and B in the most natural manner is
$A \oplus B = \{a + b \,|\, a \in A, b \in B\}$
where $a + b$ is the vector sum of two points. Convolving two LP polygons and establishing some relation involving its solutions is also an incremental result towards the study of combinatorial structure of LP polytope and a combinatorial algorithm for solving LP problem.

Key words:

INTRODUCTION

Minkowski sums commonly arise in robotics under the name configuration space obstacle for robot motion planning, in non-collision tool path generation, mathematical morphology, and this sum also plays an important role in Linear Programming problems.

A linear programming problem (LP) is to find a maximizer and minimizer of a linear function subject to linear inequality constraints, i.e.,

maximize

$$f(x) = \sum_{j=1}^{d} c_j x_j$$

subject to

$$\sum_{j=1}^{d} a_{ij} x_j \le b_i$$

A convex polygon is the intersection A of a finite number of closed halfspaces in R^2, A is a 2-dimensional polyhedron if the points in A affinely span $R^2$0. A convex 2-dimensional polytope is a bounded convex 2-polyhedron. Hence, the relevance of convex polyhedra to linear programming is clear. The set A of possible solutions for a linear programming problem is a polyhedron.

The Minkowski-Weyl theorem states that every polyhedron is finitely generated and every finitely generated set is a polyhedron, that is to say, two subsets A and B of R^d, the Minkowski sum is denoted as

$$A + B = \{a + b \mid a \in A, b \in B\}$$

where $a + b$ denotes the vector sums of the vectors a and that is to say, $a = (a_x, a_y)$ and $b = (b_x, b_y)$ then we have

$$a + b = \{a_x + b_x, a_y + b_y\}$$

Minkowski-Weyl's Theorem:

- **a)** A a polyhedron, i.e., for some real and finite matrix $[A]$ and a real vector b, we have $A = \{x : [A]x \le b\}$.
- **b)** There are finite real vectors $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_s$ in R^d such that $A = conv(v_1, v_2, \ldots, v_n) + nonneg(r_1, r_2, \ldots, r_s)$.

Corresponding Author: Ali Dino Jumani, Department of Mathematics Shah Abdul Latif University Khairpur 66020, Pakistan.

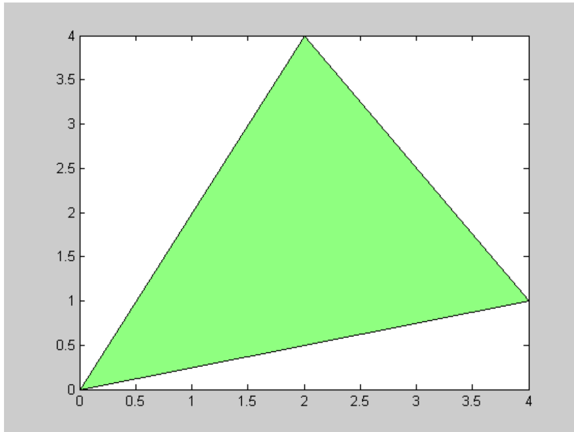

Fig. 1: LP polygon A.

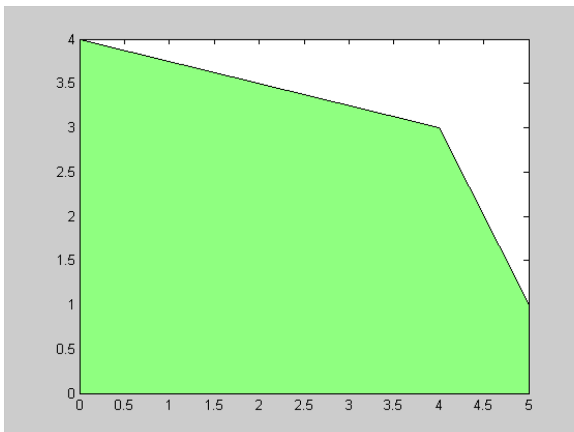

Fig. 2: LP polygon B.

Thus every polyhedron has two representations of type (a) and (b), known as *H*- representation (*halfspace*) and *V*- representation (*vertex*) where *V*- representation is also known as the convex hull. The Minkowski addition problem is to compute the set v of all extreme points of the Minkowski addition.

The boundary of the Minkowski sum of two geometric objects is part of the so-called convolution surface of the boundary surface of the two input objects.

2. Minkowski Sum

The Minkowski sum $A \oplus B$ consists of pair-wise sums of all points of A and B. Also note that from fundamental convolution, it follows that the winding number of a point with respect to $A \oplus B$ is the number of connected components in the intersection of A and B translated by a point.

Theorem 1:

Minkowski sum $A \oplus B$ of two polygons A and B with m and n edges respectively can have has at most $m+n$ edges.

Consequently the Minkowski sum of two convex polygons is a convex polygon and has linear complexity.

Theorem 2:

Resulting intersections polygons are also convex.

Proof:

Let x be a point then the Minkowski sum of x and the set B,

$$x \oplus B\{x + b \mid b \in B\}$$

this can be described as a translated copy of x, that is, each point of B is moved by x. Consequently we say that

$$A \oplus B = \bigcup_{x \in A} (x \oplus B)$$

is a union of copies of B, one for each $x \in A$. Each translation relocates the point to a newer location and forms other convex bodies.

Conversely, given two vectors a of A and b of B, their vector sum

$$a + b = (a_1, a_2) + (b_1, b_2) \qquad = (a_1 + b_1,\ a_2 + b_2) \qquad = c$$

the resulting vector c lies away from A or B. Similarly other two points a' and b'

$$a' + b' = (a_1^{'}, a_2^{'}) + (b_1^{'}, b_2^{'}) \qquad = (a_1^{'} + b_1^{'},\ a_2^{'} + b_2^{'}) \qquad = c'$$

continuing this binary operation and translating solutions will generate other polygons as in Figure 3 above.

Corollary 1:

Individual solutions of the LP polygons A and B are preserved.

Proof:

Minkowski sums, in addition to generating other polygons, preserve the original LP polygons A and B. The preservation is caused by $x \oplus (0,0)$ where x is the individual solution of each LP polygon.

Conclusion:

More striking mystery about the convex polytope, LP polytope, was posed by W. M. Hirsch in 1957. It says that the diameter of a convex polytope A is bounded by *m-n*. This makes LP much more mysterious and call for another polynomial time algorithm. Exploiting the combinatorial nature of LP polytope, summing two or more polytopes and analyzing their resulting structure point to some relations that are tempting in nature as in Theorem 1, Corollary 1.

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