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General travelling wave solutions of quintic nonlinearity of Klein-Gordon equation

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Abstract: In this letter, the (G'/G)-expansion method is proposed to construct the exact travelling solutions of quintic nonlinearity of the Klein-Gordon equation, (so called Klein-Gordon equation II), given by: $u_{tt} - p^2 u_{xx} + au - bu^3 + cu^5 = 0$, where $a, b, p \in \mathrm{R}$, $c = \frac{3b^2}{16a}$, and $ab \neq 0$. Our work is motivated by the fact that the (G'/G)-expansion method provides not only more general forms of solutions but also periodic and solitary waves. As a result, hyperbolic function solutions and trigonometric function solutions with parameters are obtained. It is shown that the proposed method is effective and can be used for many other nonlinear differential equations in mathematical physics.

Key word: quintic nonlinearity of the Klein-Gordon equation, (G'/G)-expansion method, hyperbolic function solutions, trigonometric function solutions.

INTRODUCTION

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical physical sciences such as physics, mechanics, chemistry, etc. The analytical solutions of such equations are of fundamental importance since a lot of mathematical physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as solitons. In mathematics and physics, a soliton is a self reinforcing solitary wave, a wave packet or pulse, that maintains its shape while it travels at constant speed. Solitons are caused by a cancelation of nonlinear and dispersive effects in the medium. The term "dispersive effects" refers to a property of certain systems where the speed of the waves varies according to frequency. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. The soliton phenomenon was first described by *John Scott Russell* (1808–1882) who observed a solitary wave in the Union Canal in Scotland. He reproduced the phenomenon in a wave tank and named it the "Wave of Translation" (also known as travelling wave solutions or solitons)(Scott Russell, 1844) The soliton solutions are typically obtained by means of the inverse scattering transform (Ablowitz, and Segur, 1981) and owe their stability to the integrability of the field equations.

In the literature, there is a wide variety of approaches to nonlinear problems for constructing travelling wave solutions, such as the inverse scattering method (Ablowitz, and Segur, 1981) Bäcklund transformation (Tam, and Hu, 2002) Hirota bilinear method (Hirota, 2004) numerical methods (Reza Abazari, and Borhanifar, 2010) and the Wronskian determinant technique (Freeman, and Nimmo, 1983)

With the help of the computer software, most of mentioned methods are improved and many other algebraic method proposed, such as the Jacobi elliptic function method (Liu, and Fan, 2005) homogeneous balance method (Wang, 1996) perturbation method (Shabani Shahrbabaki, and Reza Abazari, 2009) homotopy perturbation method (He, 2004) differential transform method (Borhanifar, and Reza Abazari, 2010) (Reza Abazari, and Masoud Ganji, 2011) Adomian decomposition method (Adomian, 1994) sine/cosine method (Yan, 1996) tanh/coth method (Malfliet, and Hereman, 1996) the Exp--function method (Fei Xu, 2008) first integral method (Filiz Tascan, and Ahmet Bekir, 2009) and a new method (Wu Guo-cheng, Xia Tie-cheng, 2008) But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of nonlinearity increases.

Recently, the (G'/G)-expansion method, firstly introduced by Wang *et al*. 2008: Tan, 1996) has become widely used to search for various exact solutions of NLEEs (Wang *et al*. 2008: Kabir, *et al*., 2011) Although many efforts have been devoted to find various methods to solve (integrable or non-integrable) NLEEs, there is no a unified method. The main merits of the (G'/G)-expansion method over the other methods are that it gives more general solutions with some free parameters which, by suitable choice of the parameters, turn out to be some known solutions gained by the existing methods. Besides, (i) in all finite difference and finite element methods, it is necessary to have boundary and initial conditions. However, the (G'/G)-expansion method

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handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. It obtains a general solution with free parameters that can be determined via boundary and/or initial conditions, (ii) most of the methods give solutions in a series form and it becomes essential to investigate the convergence of approximation series. For example, the Adomian decomposition method depends only on the initial conditions and obtains a solution in a series which converges to the exact solution of the problem. But, with the (G'/G)-expansion method, one may obtain a general solution without approximation, (iii) it serves as a powerful technique to integrate the NLEEs, even if the Painleve test of integrability fails, (iv) the solution procedure, using a computer algebra system like Mathematica, is of utter simplicity.

The main idea of this method is that the traveling wave solutions of nonlinear equations can be expressed by a polynomial in (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{1.1}$$

where $\xi = kx + \omega t$ and k, ω are arbitrary constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and the non--linear terms appearing in the given nonlinear equations. For more nonlinear equations, the degree of this polynomial is a positive integer. But there are some nonlinear PDEs where this degree is fractional, such as the quintic nonlinearity of the Klein-Gordon equation, (so called Klein-Gordon equation II), given by:

$$u_{tt} - p^2 u_{xx} + au - bu^3 + cu^5 = 0, \tag{1.2}$$

where $a, b, p \in \mathrm{R}$, $c = \dfrac{3b^2}{16a}$, and $ab \neq 0.$ The Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons and condensed matter physics (Caudrey, *et al*., 1975) in investigating the interaction of solitons in a collision less plasma, the recurrence of initial states, and in examining the nonlinear wave equations (Dodd, *et al*., 1982)

The cubic form of Eq. (1.1) models a variety of physical phenomena. Due to the wide applications of these equations, many solutions have been obtained in different functional forms by different methods (Wazwaz, 2006: Ryan Sassaman, and Anjan Biswas, 2009: Ryan Sassaman, Anjan Biswas, 2009) But in quintic form, a small amount of work has been done (see, for example (Wazwaz, 2006: Bratsos, L. A. Petrakis, 2010).

Our first interest in the present work is in implementing the (G'/G)-expansion method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact traveling wave solutions for the Klein-Gordon equation II (1.1).

1 Description of the (G'/G)-expansion method

The objective of this section is to outline the use of the (G'/G)-expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_x, u_t, u_{xx}, u_{x,t}, u_{tt}, ...) = 0, \tag{2.1}$$

where $u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. The main steps of the (G'/G)-expansion method are the following:

Step 1. Combining the independent variables x and t into one variable $\xi = kx + \omega t,$ we suppose that

$$u(x,t) = U(\xi), \tag{2.2}$$

The travelling wave variable (2.2) permits us to reduce Eq. (2.1) to an ODE for $u(x,t) = U(\xi)$, namely

$$P(U, kU', \omega U', k^2U'', k\omega U'', \omega^2 U'', ...) = 0, \tag{2.3}$$

where prime denotes derivative with respect to ξ.

Step 2. We assume that the solution of Eq. (2.3) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = \sum_{i=1}^{m} \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \tag{2.4}$$

where $\alpha_m \neq 0$ and m is called the balance number, α_0, and $\alpha_i, (i = 1, 2, ...m)$ are constants to be determined later, $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0. \tag{2.5}$$

where λ and μ are arbitrary constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (2.3).

Step 3. By substituting (2.4) into Eq. (2.3) and using the second order linear ODE (2.5), collecting all terms with the same order of (G'/G) together, the left-hand side of Eq. (2.3) is converted into another polynomial in (G'/G). Equating each coefficient of this polynomial to zero yields a set of algebraic equations for

$$k, \omega, \lambda, \mu, \alpha_0, \alpha_1, ..., \alpha_m.$$

Step 4. Assuming that the constants $k, \omega, \lambda, \mu, \alpha_0, \alpha_1, ..., \alpha_m$ can be obtained by solving the algebraic equations in **Step 3**, since the general solutions of the second order linear ODE (2.5) is well known for us,

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \dfrac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\dfrac{C_1 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_2 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1 \cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + C_2 \sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)} \right) - \dfrac{\lambda}{2}, & \text{for } \lambda^2 - 4\mu > 0, \\ \dfrac{\sqrt{4\mu - \lambda^2}}{2} \left(\dfrac{-C_1 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_2 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1 \cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + C_2 \sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)} \right) - \dfrac{\lambda}{2}, & \text{for } \lambda^2 - 4\mu < 0, \end{cases} \tag{2.6}$$

then substituting $k, \omega, \lambda, \mu, \alpha_0, ..., \alpha_m$ and the general solutions (2.6) into (2.4) we have more travelling wave solutions of the nonlinear evolution Eq. (2.1).

***Remark 2.1*:**

On comparing between the (G'/G)-expansion method and the modified tanh-coth method [27], we first summarize the last method as follows:

Suppose the solution of Eq. (2.3) in a finite series of function of the form:

$$u(\xi) = \alpha_0 + \sum_{\ell=1}^{m} \{\alpha_\ell Y^{\ell}(\xi) + \beta_\ell Y^{-\ell}(\xi)\}, \tag{2.7}$$

where $Y(\xi)$ satisfies the Riccati equation.

$$\frac{dY(\xi)}{d\xi^2} + Y^2(\xi) + \lambda Y(\xi) + \mu = 0, \tag{2.8}$$

The parameter "m" can be determined by the homogeneous balance. Inserting (2.7) into (2.3) and using (2.8) results a set of algebraic equations which can be solved to determine $k, \omega, \alpha_\ell, \beta_\ell, \lambda, \mu$ for $\ell = 1, 2, ..., m$

where the general solution of (2.8) is well known. Having these parameters we obtain an analytic solution. $u(x,t)$ in a closed form. If we put $\beta_\ell = 0,$ and $Y(\xi) = \dfrac{G'(\xi)}{G(\xi)}$ in (2.7) and (2.8), then we get immediately (2.4) and (2.5), respectively. This shows that the (G'/G)-expansion method is more effective and convenient than the modified tanh-coth method for the following reasons:

1- In the case of the (G'/G)-expansion method we use the solutions of the second order linear ODE (2.5) which are easier than the solutions of the Riccati Eq. (2.8).

2- The exact solutions of Eq. (2.1) using the (G'/G)-expansion method contain more arbitrary constants compared to the exact solutions presented by the modified tanh-coth method.

Remark 2.2:

Now to comparing between the (G'/G)-expansion method and the Exp-function method, we use the Exp-function method to deal with Eq. (1.1) and can obtain the following two solutions:

$$G(\xi) = \frac{A_0}{B_0} e^{\frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}\xi}, \tag{2.9}$$

$$G(\xi) = \frac{A_0}{B_0} e^{\frac{-\lambda - \sqrt{\lambda^2 - 4\mu}}{2}\xi}, \tag{2.10}$$

where A_0 and B_0 are free parameters. Because Eq. (1.1) is a linear equation, the linear combination of solutions (2.9) and (2.10) is its general solution:

$$G(\xi) = \bar{C}_1 \frac{A_0}{B_0} e^{\frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}\xi} + \bar{C}_2 \frac{A_0}{B_0} e^{\frac{-\lambda - \sqrt{\lambda^2 - 4\mu}}{2}\xi}, \tag{2.11}$$

where $\bar{C}_1$ and $\bar{C}_2$ are arbitrary constants. We therefore have.

$$\frac{G'}{G} = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{\bar{C}_1 e^{\frac{\sqrt{\lambda^2-4\mu}}{2}\xi} - \bar{C}_2 e^{\frac{-\sqrt{\lambda^2-4\mu}}{2}\xi}}{\bar{C}_1 e^{\frac{\sqrt{\lambda^2-4\mu}}{2}\xi} + \bar{C}_2 e^{\frac{-\sqrt{\lambda^2-4\mu}}{2}\xi}}\right) - \frac{\lambda}{2}, \tag{2.12}$$

further setting $\bar{C}_1 = C_1 + C_2$ and $\bar{C}_2 = C_1 - C_2,$ and letting $\lambda^2 - 4\mu > 0,$ we obtain

$$\frac{G'}{G} = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}\right) - \frac{\lambda}{2}, \tag{2.13}$$

where C_1 and C_2 are arbitrary constants. If set $\bar{C}_1 = C_1 - iC_2$ and $\bar{C}_2 = C_1 + iC_2,$ here $i \equiv \sqrt{-1}$ and let $\lambda^2 - 4\mu < 0,$ we also obtain

$$\frac{G'}{G} = \frac{\sqrt{4\mu - \lambda^2}}{2}\left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}\right) - \frac{\lambda}{2}, \tag{2.14}$$

Employing Eq. (1.1) and solutions (2.13) and (2.14), and using the Exp-function method, we can obtain solutions same as the solutions of $(\frac{G'}{G})$-expansion method.

Remark 2.3:

There are some new developments of the (G'/G)-expansion method listed as follows:

1- The authors in (Zhang, *et al*., 2008) consider (2.4) where α_ℓ are functions and $G = G(\xi)$ satisfies (2.5).

2- The authors in (Sheng Zhang, *et al*., 2009) consider a difference form of (2.4) where α_ℓ are constants and $G = G(\xi)$ satisfies (2.5) to obtain the travelling wave solutions of nonlinear differential-difference equations.

The new developments (I) and (II) will be discussed in forthcoming articles for further nonlinear evolution equations in mathematical physics.

In the subsequent section, we will illustrate the validity and reliability of this method in detail with quintic nonlinearity of Klein-Gordon equation (1.1) where the balance numbers of which is not positive integer.

2 Application

To look for the traveling wave solution of quintic nonlinearity of Klein--Gordon equation (1.2), we use the gauge transformation:

$$u(x,t) = U(\xi), \tag{3.1}$$

where $\xi = kx + \omega t,$ and k, ω are constants. We substitute Eq. (3.1) into Eq. (1.2) to obtain nonlinear ordinary differential equation.

$$(\omega^2 - p^2k^2)U'' + aU - bU^3 + cU^5 = 0, \tag{3.2}$$

According to *Step 1*, we get $m + 2 = 5m$, hence $m = \frac{1}{2}$. We then suppose that Eq. (3.2) has the following formal solutions:

$$U = \alpha_1 (\frac{G'}{G})^{\frac{1}{2}}, \qquad \alpha_1 \neq 0, \tag{3.3}$$

where $\alpha_1,$ is constants which are unknown to be determined later. Therefore,

$$U' = -\frac{1}{2}\alpha_1(\frac{G'}{G})^{\frac{3}{2}} - \frac{1}{2}\alpha_1\lambda(\frac{G'}{G})^{\frac{1}{2}} - \frac{1}{2}\alpha_1\mu(\frac{G'}{G})^{-\frac{1}{2}}, \tag{3.4}$$

$$U'' = \frac{3}{4}\alpha_1(\frac{G'}{G})^{\frac{5}{2}} + \alpha_1\lambda(\frac{G'}{G})^{\frac{3}{2}} + \frac{1}{2}\alpha_1(\mu + \frac{1}{2}\lambda^2)(\frac{G'}{G})^{\frac{1}{2}} - \frac{1}{4}\alpha_1\mu^2(\frac{G'}{G})^{-\frac{3}{2}}, \tag{3.5}$$

Substituting Eqs. (3.3)-(3.5) into Eq. (3.2) and collecting all terms with the same order of $(\frac{G'}{G})$ together, the left-hand sides of Eq. (3.2) are converted into a polynomial in (G'/G). Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for k, ω, λ, μ and α_1 as follows:

$$p^2(2\mu + \lambda^2)k^2 - (2\omega^2\mu + \omega^2\lambda^2 + 4a) = 0, \tag{3.6}$$

$$(p^2k^2 - \omega^2)\lambda + b\alpha_1^2 = 0, \tag{3.7}$$

$$4a(p^2k^2 - \omega^2) - b^2\alpha_1^4 = 0, \tag{3.8}$$

Solving (3.6)-(3.8), simultaneously, by use of Maple, we get the following results:

$$\text{case I}: \quad \{\lambda = \frac{-b\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}}{p^2k^2-\omega^2}, \mu = 0, \alpha_1 = \pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\} \tag{3.9}$$

And

$$\text{case II}: \quad \{\lambda = \frac{b\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}}{p^2k^2-\omega^2}, \mu = 0, \alpha_1 = \pm i(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\} \tag{3.10}$$

and k, ω are free constant parameters.

Case I:

On substituting the case (3.9) in (3.3), we get

$$U = \pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}(\frac{G'}{G})^{\frac{1}{2}}, \tag{3.11}$$

Substituting the general solutions of ordinary differential equation (2.5) into Eq. (3.11), we obtain two types of traveling wave solutions of Eq. (1.2) in view of the positive and negative of $\lambda^2 - 4\mu$. When $\mathsf{D} = \lambda^2 - 4\mu = \frac{a}{p^2k^2-\omega^2} > 0,$ using the general solutions of ordinary differential equation (2.5), we obtain hyperbolic function solution U_{H} of quintic nonlinearity of Klein--Gordon equation (1.2) as follows:

$$U_{\mathsf{H}}(\xi) = \pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}} \quad [\frac{\sqrt{\mathsf{D}}}{2}(\frac{C_1\sinh(\frac{\sqrt{\mathsf{D}}}{2}\xi) + C_2\cosh(\frac{\sqrt{\mathsf{D}}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\mathsf{D}}}{2}\xi) + C_2\sinh(\frac{\sqrt{\mathsf{D}}}{2}\xi)}) + \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.12}$$

where $\xi = kx + \omega t,$ and $C_1, C_2,$ are arbitrary constants. It is easy to see that the hyperbolic solution (3.12) can be rewritten at $C_1^2 > C_2^2,$ as follows

$$u_{\mathsf{H}}(x,t) = \pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}} \quad [\frac{\sqrt{\mathsf{D}}}{2}\tanh(\frac{\sqrt{\mathsf{D}}}{2}\xi + \eta_{\mathsf{H}}) + \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.13}$$

while at $C_1^2 < C_2^2,$ one can obtain

$$u_{\mathsf{H}}(x,t) = \pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}} \quad [\frac{\sqrt{\mathsf{D}}}{2}\coth(\frac{\sqrt{\mathsf{D}}}{2}\xi + \eta_{\mathsf{H}}) + \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.14}$$

where $\xi = kx + \omega t, \eta_{\mathsf{H}} = \tanh^{-1}(\frac{C_1}{C_2}),$ and $k, \omega,$ are arbitrary constants.

Now, when $\mathsf{D} = \lambda^2 - 4\mu = \frac{4a}{a^2k^2-\omega^2} < 0,$ we obtain trigonometric function solution $U_{\mathsf{T}},$ of Eq. (1.2) as follows:

$$U_{T}(\xi)=\pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{-D}}{2}(\frac{-C_1\sin(\frac{\sqrt{-D}}{2}\xi)+C_2\cos(\frac{\sqrt{-D}}{2}\xi)}{C_1\cos(\frac{\sqrt{-D}}{2}\xi)+C_2\sin(\frac{\sqrt{-D}}{2}\xi)})+\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.15}$$

where $\xi = kx+\omega t,$ and $C_1, C_2,$ are arbitrary constants. Similarity, it is easy to see that the trigonometric solution (3.15) can be rewritten at $C_1^2 > C_2^2,$ and $C_1^2 < C_2^2,$ as follows

$$u_{T}(x,t)=\pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{-D}}{2}\tan(\frac{\sqrt{-D}}{2}\xi+\eta_{T})+\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.16}$$

And

$$u_{T}(x,t)=\pm(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{-D}}{2}\cot(\frac{\sqrt{-D}}{2}\xi+\eta_{T})+\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.17}$$

respectively, where $\xi = kx+\omega t, \eta_{T} = \tan^{-1}(\frac{C_1}{C_2}),$ and $k, \omega,$ are arbitrary constants.

Case II:

Similar on caseI, on substituting the case (3.10) in (3.3), we get

$$U=\pm i(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}(\frac{G'}{G})^{\frac{1}{2}}, \tag{3.18}$$

Therefore, when $D=\lambda^2-4\mu=\frac{a}{p^2k^2-\omega^2}>0,$ the hyperbolic function solution U_{H} of quintic nonlinearity of Klein--Gordon equation (1.2) obtained as follows:

$$U_{H}(\xi)=\pm i(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{D}}{2}(\frac{C_1\sinh(\frac{\sqrt{D}}{2}\xi)+C_2\cosh(\frac{\sqrt{D}}{2}\xi)}{C_1\cosh(\frac{\sqrt{D}}{2}\xi)+C_2\sinh(\frac{\sqrt{D}}{2}\xi)})-\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.19}$$

where $\xi = kx+\omega t,$ and $C_1, C_2,$ are arbitrary constants. Similarly, the hyperbolic solution (3.19) can be rewritten at $C_1^2 > C_2^2,$ and $C_1^2 < C_2^2,$ as follows

$$u_{H}(x,t)=\pm i(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{D}}{2}\tanh(\frac{\sqrt{D}}{2}\xi+\eta_{H})-\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.20}$$

$$u_{H}(x,t)=\pm i(\frac{4a(p^2k^2-\omega^2)}{b^2})^{\frac{1}{4}}\quad[\frac{\sqrt{D}}{2}\coth(\frac{\sqrt{D}}{2}\xi+\eta_{H})-\frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}]^{\frac{1}{2}}, \tag{3.21}$$

where $\xi = kx+\omega t, \eta_{H} = \tanh^{-1}(\frac{C_1}{C_2}),$ and $k, \omega,$ are arbitrary constants. Now, when

$\Delta = \lambda^2 - 4\mu = \frac{4a}{a^2k^2 - \omega^2} < 0,$ we obtain trigonometric function solution U_{T}, of Eq. (1.2) as follows:

$$U_{\mathrm{T}}(\xi) = \pm i\left(\frac{4a(p^2k^2-\omega^2)}{b^2}\right)^{\frac{1}{4}} \left[\frac{\sqrt{-\Delta}}{2}\left(\frac{-C_1\sin(\frac{\sqrt{-\Delta}}{2}\xi) + C_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}{C_1\cos(\frac{\sqrt{-\Delta}}{2}\xi) + C_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}\right) - \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}\right]^{\frac{1}{2}}, \tag{3.22}$$

where can be rewritten at $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows

$$u_{\mathrm{T}}(x,t) = \pm i\left(\frac{4a(p^2k^2-\omega^2)}{b^2}\right)^{\frac{1}{4}} \left[\frac{\sqrt{-\Delta}}{2}\tan\left(\frac{\sqrt{-\Delta}}{2}\xi + \eta_{\mathrm{T}}\right) - \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}\right]^{\frac{1}{2}}, \tag{3.23}$$

$$u_{\mathrm{T}}(x,t) = \pm i\left(\frac{4a(p^2k^2-\omega^2)}{b^2}\right)^{\frac{1}{4}} \left[\frac{\sqrt{-\Delta}}{2}\cot\left(\frac{\sqrt{-\Delta}}{2}\xi + \eta_{\mathrm{T}}\right) - \frac{1}{2}\frac{b}{p^2k^2-\omega^2}\sqrt{\frac{4a(p^2k^2-\omega^2)}{b^2}}\right]^{\frac{1}{2}}, \tag{3.24}$$

respectively, where $\xi = kx + \omega t, \eta_{\mathrm{T}} = \tan^{-1}(\frac{C_1}{C_2}),$ and $k, \omega,$ are arbitrary constants.

Conclusions:

This study shows that the (G'/G)-expansion method is quite efficient and practically well suited for use in finding exact solutions for the quintic nonlinearity of Klein-Gordon equation where the balance numbers of which is not positive integer. Our solutions are in more general forms, and many known solutions to these equations are only special cases of them. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equation. On comparing between the (G'/G)-expansion method and the other methods such as the modified tanh-coth method and Exp-function method, we conclude that the (G'/G)-expansion method is similar on these methods. The performance of these method are reliable, simple and gives many new exact solutions. However, the (G'/G)-expansion method has more advantages, because the hyperbolic function and trigonometric function solutions of (G'/G)-expansion method obtained directly and concisely. It is also a standard and computerizable method which allows us to solve complicated nonlinear evolution equations in mathematical physics. We have noted that the (G'/G)-expansion method changes the given difficult problems into simple problems which can be solved easily.

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