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Study of Deterministic Systems in Epidemic Diseases

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Abstract: This article presents the deterministic system for susceptible-infective model for HIV. In this paper the homotopy analysis method is employed to compute an exact analytical approximation to the solution of the deterministic system for this model. We do a comparison between this method and Runge-Kutta method.

Key words: Homotopy analysis method; Susceptible-Infective model; Runge-Kutta method.

INTRODUCTION

We considered the basic SI model for HIV as,

$$\begin{cases} \frac{dS(t)}{dt} = \mu(S^0 - S(t)) - \Upsilon S(t)I(t), \\ \frac{dI(t)}{dt} = \Upsilon S(t)I(t) - \mu I(t), \end{cases} \tag{1}$$

subject to initial conditions $I(0) = I_0, S(0) = S_0$ where μS^0 is the rate of population as new susceptible into class $S, \Upsilon = \frac{r\beta}{N}$; S+I=N, where N is total population s ize (Hethcote, 2000). The disease-free equilibrium, reproductive number and endemic equilibrium for this model are at order $E_0 = (S^0, 0), R_0 = \frac{r\beta}{\mu}$ and $E_e = (\frac{S^0}{R_0}, \frac{R_0 - 1}{R_0} S^0)$. So we have two different cases for t approaching ∞,

(*i*) If $R_0 = \frac{r\beta}{\mu} \le 1$ for any I_0, then,

$I(+\infty) = 0, S(+\infty) = S^0,$

(*ii*) If $R_0 = \frac{r\beta}{\mu} > 1$ for any I_0, then,

$I(+\infty) = S^0(1 - \frac{\mu}{r\beta}), S(+\infty) = \frac{\mu S^0}{r\beta},$

Analysis of method:

We will first present a brief description of the standard Homotopy Analysis Method (HAM) (Liao, 2004; Liao, 1992; Liao, 2009). This will be followed by a description of the algorithm of the Modified Homotopy Analysis Method (MHAM). To achieve our goal, we consider the differential equation

$$N[v(t)] = g(t) \tag{2}$$

Where *N* are nonlinear operators, denotes the independent variable, *v*(*t*) are unknown functions and *g*(*t*) are known analytic functions. For *g*=0, Eq. (2) reduces to the homogeneous equation. By means of generalizing the traditional homotopy method, Liao (Liao, 2003) constructs the so-called zeroth-order deformation equation

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$$(1-p)L[\Psi(t;p)-v_0(t)] = p\hbar\{[N(\Psi(t;p)]-g(t)\}, \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ are nonzero auxiliary functions, L is an auxiliary linear operator, $v_0(t)$ are initial guesses of $v(t)$ and $\Psi(t;p)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary object such as $\hbar$ and L in HAM. Obviously, when p=0 and p=1, both

$$v_0(t) = \Psi(t;0) - v_0(t) \qquad \text{and} \qquad v(t) = \Psi(t;1),$$

hold. Thus as p increases from 0 to 1, the solutions $\Psi(t;p)$ varies from the initial guesses $v_0(t)$ to the solutions $v(t)$. Expanding $\Psi(t;p)$ in Taylor series with respect to p, one has

$$\Psi(t;p) = v_0(t) + \sum_{m=1}^{+\infty} v_m(t)p^m, \tag{4}$$

where

$$v_m = \frac{1}{m!}\frac{\partial^m \Psi(t;p)}{\partial p^m}\Big|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameters $\hbar$, and the auxiliary functions are so properly chosen, then the series (4) converges at p=1, and one has

$$\Psi(t;1) = v_0(t) + \sum_{m=1}^{+\infty} v_m(t),$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao (Liao, 2005). As $\hbar$=-1, Eq. (3) becomes

$$(1-p)L[\Psi(t;p)-v_0(t)] + p\{[N(\Psi(t;p)]-g(t)\} = 0, \tag{6}$$

which is used mostly in the homotopy perturbation method (HPM) (Liao, 2000). According to Eq. (4), the governing equations can be deduced from the *zeroth-order deformation equation* (3). Define the vectors

$$\vec{v}_n = \{v_0(t), v_1(t), v_2(t), \ldots, v_n(t)\}. \tag{7}$$

Differentiating Eq. (3) m times with respect to the embedding parameter p and then setting p=0 and finally dividing by $m!$, we have the so-called *mth-order equation*

$$L[v_m(t) - \chi_m v_{m-1}(t)] = \hbar R_m(\vec{v}_{m-1}), \tag{8}$$

where

$$R_m(\vec{v}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\{N[\Psi(t;q)]-g(t)\}}{\partial p^{m-1}}\Big|_{p=0}, \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $v_m(t)(m \ge 1)$ is governed by the linear equation (8) with the linear boundary conditions that come from the original problem.

Zeroth-order Deformation Equations:

To solve Eq. (1) by means of homotopy analysis method (Liao 2009) we chose the nonlinear operator,

$$\aleph_S[S(t;p),I(t;p)] = \frac{\partial S(t;p)}{\partial t} - \mu S^0 + \mu S(t;p) + \Upsilon S(t;p)I(t;p),$$
$$\aleph_I[S(t;p),I(t;p)] = \frac{\partial I(t;p)}{\partial t} - \Upsilon S(t;p)I(t;p) + \mu I(t;p), \tag{10}$$

Let $S_0(t), I_0(t)$ denote the initial guesses of $S(t)$ and $I(t)$, $\pounds_S$ and $\pounds_I$ the two auxiliary linear operators, $H_S(t)$ and $H_I(t)$ the two non-zero auxiliary functions, and $\hbar$ a non-zero auxiliary parameter, called the convergence control parameter. We will determine all of them later. We have great freedom to choose all of them. Let $p \in [0,1]$ denote the embedding parameter. Then we construct the family of the differential equations,

$$(1-p)\pounds_S[S(t;p) - S_0(t)] = p\hbar H_S(t)\aleph_S[S(t;p),I(t;p)], \tag{11}$$

$$(1-p)\pounds_S[I(t;p) - I_0(t)] = p\hbar H_I(t)\aleph_S[S(t;p),I(t;p)], \tag{12}$$

with the initial conditions,

$$S(0;p) = S_0, \quad I(0;p) = I_0. \tag{13}$$

Obviously, when p=0 and p=1, it holds,

$S(0;p) = S_0(t), I(0;p) = I_0(t)$and $S(1;p) = S(t), I(1;p) = I(t)$.

Thus as p increase 0 to 1, the solutions $S(t;p)$ and $I(t;p)$ varies from the initial guesses $S_0(t)$ and $I_0(t)$ to the solutions $S(t)$ and $I(t)$, respectively. Expanding $S(t;p)$ and $I(t;p)$ in the Taylor series with respect to p, one has,

$$S(t;p) = S_0(t) + \sum_{m=1}^{+\infty} S_m(t)p^m, \tag{14}$$

$$I(t;p) = I_0(t) + \sum_{m=1}^{+\infty} I_m(t)p^m, \tag{15}$$

where

$$S_m = \frac{1}{m!}\frac{\partial^m S(t;p)}{\partial p^m}\bigg|_{p=0}, \tag{16}$$

$$I_m = \frac{1}{m!}\frac{\partial^m I(t;p)}{\partial p^m}\bigg|_{p=0}. \tag{17}$$

If the auxiliary linear operators, the initial guesses, and the auxiliary parameter $\hbar$ are so properly chosen, the series (14) and (15) are converge at p=1, one has,

$$I_m = \frac{1}{m!}\frac{\partial^m I(t;p)}{\partial p^m}\bigg|_{p=0}. \tag{18}$$

$$I(t) = I_0(t) + \sum_{m=1}^{+\infty} I_m(t), \tag{19}$$

High-order Deformation Equation:

We define the vectors,

$$\vec{S}_m = \{S_0(t), S_1(t), ..., S_m(t)\}, \tag{20}$$

$$\vec{I}_m = \{I_0(t), I_1(t), ..., I_m(t)\}. \tag{21}$$

Differentiating Eq. (11) and Eq. (12) m times with respect to the embedding parameter p and then setting p=0 and finally dividing them by $m!$, we have the so-called m th-order deformation equations,

$$\pounds_S[S_m(t) - \chi_m S_{m-1}(t)] = \hbar H_S(t) \Re_m^S(t), \tag{22}$$

$$\pounds_I[I_m(t) - \chi_m I_{m-1}(t)] = \hbar H_I(t) \Re_m^I(t), \tag{23}$$

with the initial conditions,

$$S_m(0) = 0, \quad I_m(0) = 0, \tag{24}$$

where

$$\Re_m^S(t) = \frac{dS_{m-1}(t)}{dt} - \mu S^0 + \mu S_{m-1}(t) + \Upsilon \sum_{k=0}^{m-1} I_k(t) S_{m-1-k}(t), \tag{25}$$

$$\Re_m^I(t) = \frac{dI_{m-1}(t)}{dt} + \mu I_{m-1}(t) - \Upsilon \sum_{k=0}^{m-1} I_k(t) S_{m-1-k}(t), \tag{26}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $S_m(t)$ and I_m for $m \ge 1$ are governed by the linear equations (22) and (23) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab, Maple and Mathematica.

Explicit Series Solution of the Deterministic Si Model:

Since $S(t) \to S_\infty$ and $I(t) \to I_\infty$ as $t \to \infty$, So $S(t)$ and $I(t)$ can be expressed by,

$$S(t) = S_\infty + \sum_{k=1}^{+\infty} a_k e^{-k\alpha t}, \tag{27}$$

$$I(t) = I_\infty + \sum_{k=1}^{+\infty} b_k e^{-k\alpha t}, \tag{28}$$

where a_k and b_k are coefficients to be determined. From Eqs. (1), we have,

$$S'(0) = \mu S^0 - \mu S(0) - \Upsilon S(0) I(0), \tag{29}$$

$$I'(0) = \Upsilon S(0) I(0) - \mu I(0), \tag{30}$$

where, $\Upsilon = \frac{r\beta}{N(0)}$. To obey the solution expressions (27) and (28), we choose the initial guesses $S_0(_t)$ and $I_0(t)$ such that,

$$S_0(t) = S_\infty + \zeta_{0,1} e^{-\alpha t} + \zeta_{0,2} e^{-2\alpha t} + \zeta_{0,3} e^{-3\alpha t}, \tag{31}$$

$$I_0(t) = I_\infty + \gamma_{0,1} e^{-\alpha t} + \gamma_{0,2} e^{-2\alpha t} + \gamma_{0,3} e^{-3\alpha t}, \tag{32}$$

where,

$$\zeta_{0,1} = \zeta_{0,1},$$

$$\zeta_{0,2} = 3S_0 - 3S_\infty + \frac{1}{\alpha}(\mu S^0 - \mu S_0 - \Upsilon S_0 I_0) - 2\zeta_{0,1},$$

$$\zeta_{0,3} = 2S_\infty - 2S_0 + \frac{1}{\alpha}(\mu S^0 - \mu S_0 - \Upsilon S_0 I_0) + \zeta_{0,1},$$

$$\gamma_{0,1} = \gamma_{0,1},$$

$$\gamma_{0,2} = 3I_0 - 3I_\infty + \frac{1}{\alpha}(\mu I_0 - \Upsilon S_0 I_0) - 2\gamma_{0,1},$$

$$\gamma_{0,3} = 2I_\infty - 2I_0 + \frac{1}{\alpha}(\mu I_0 - \Upsilon S_0 I_0) + \gamma_{0,1}$$

To obtain solutions in the form of Eq. (27) and Eq. (28), we choose the auxiliary linear operators,

$$£_S[S(t;p)] = \frac{\partial S(t;p)}{\partial t} + \alpha S(t;p), \tag{33}$$

$$£_I[I(t;p)] = \frac{\partial I(t;p)}{\partial t} + \alpha I(t;p), \tag{34}$$

with the property,

$$£_S[c_1 e^{-\alpha t}] = 0, \quad £_I[c_2 e^{-\alpha t}] = 0, \tag{35}$$

where c_1 and c_2 are the integral constants. Substituting the initial guesses $I_0(t)$ and $S_0(t)$ into Eq. (25) and Eq. (26), we get,

$$\Re_1^S(t) = \sum_{k=0}^{6} a_{1,k} e^{-k\alpha t}, \quad \Re_1^I(t) = \sum_{k=0}^{6} b_{1,k} e^{-k\alpha t}, \tag{36}$$

where,

$$a_{1,0} = \mu S_\infty - \mu S^0 + S_\infty I_\infty \Upsilon,$$

$$a_{1,1} = \mu\zeta_{0,1} - \alpha\zeta_{0,1} + \gamma_{0,1} S_\infty \Upsilon + \zeta_{0,1} I_\infty \Upsilon,$$

$$a_{1,2} = \mu\zeta_{0,2} - 2\alpha\zeta_{0,2} + \zeta_{0,1}\gamma_{0,1}\Upsilon + \gamma_{0,2} S_\infty \Upsilon + \zeta_{0,2} I_\infty \Upsilon,$$

$$a_{1,3} = \mu\zeta_{0,3} - 3\alpha\zeta_{0,3} + \zeta_{0,2}\gamma_{0,1}\Upsilon + \zeta_{0,1}\gamma_{0,2}\Upsilon + \gamma_{0,3} S_\infty \Upsilon + \zeta_{0,3} I_\infty \Upsilon,$$

$$a_{1,4} = \zeta_{0,3}\gamma_{0,1}\Upsilon + \zeta_{0,2}\gamma_{0,2}\Upsilon + +\zeta_{0,1}\gamma_{0,3}\Upsilon,$$

$$a_{1,5} = \zeta_{0,3}\gamma_{0,2}\Upsilon + \zeta_{0,2}\gamma_{0,3}\Upsilon,$$

$$a_{1,6} = \zeta_{0,3}\gamma_{0,3}\Upsilon,$$

$$b_{1,0} = \mu I_\infty - S_\infty I_\infty \Upsilon,$$

$$b_{1,1} = \mu\gamma_{0,1} - \alpha\gamma_{0,1} - \gamma_{0,1} S_\infty \Upsilon - \zeta_{0,1} I_\infty \Upsilon,$$

$$b_{1,2} = \mu\gamma_{0,2} - 2\alpha\gamma_{0,2} - \zeta_{0,1}\gamma_{0,1}\Upsilon - \gamma_{0,2} S_\infty \Upsilon - \zeta_{0,2} I_\infty \Upsilon,$$

$$b_{1,3} = \mu\gamma_{0,3} - 3\alpha\gamma_{0,3} - \zeta_{0,2}\gamma_{0,1}\Upsilon - \zeta_{0,1}\gamma_{0,2}\Upsilon - \gamma_{0,3} S_\infty \Upsilon - \zeta_{0,3} I_\infty \Upsilon,$$

$$b_{1,4} = -\zeta_{0,3}\gamma_{0,1}\Upsilon - \zeta_{0,2}\gamma_{0,2}\Upsilon - \zeta_{0,1}\gamma_{0,3}\Upsilon,$$

$$b_{1,5} = -\zeta_{0,3}\gamma_{0,2}\Upsilon - \zeta_{0,2}\gamma_{0,3}\Upsilon,$$

$$b_{1,6} = -\zeta_{0,3}\gamma_{0,3}\Upsilon,$$

are coefficient. We know that

$$u'(t) + \alpha u(t) = Ae^{-\mu t} + B, \tag{37}$$

has the general solution,

$$u(t) = Ate^{-\alpha t} + C_1 e^{-\alpha t} + \frac{B}{\alpha}, \tag{38}$$

where C_1 is an integral constant. Obviously, the term te^{-at} does not satisfy the expressions (27) and (28). Fortunately, we have freedom to choose the auxiliary functions $H_S(t)$ and $H_I(t)$, and thus we can avoid the appearance of the term te^{-at} simply by means of choosing,

$$H_S(t) = e^{-2\alpha t}, \;\; H_I(t) = e^{-2\alpha t} \;. \tag{39}$$

Then the first-order deformation equations become,

$$S_1'(t) + \alpha S_1(t) = \hbar \sum_{k=0}^{6} a_{1,k} e^{-(k+2)\alpha t}, \quad S_1(0) = 0, \tag{40}$$

$$I_1'(t) + \alpha I_1(t) = \hbar \sum_{k=0}^{6} b_{1,k} e^{-(k+2)\alpha t}, \quad I_1(0) = 0, \tag{41}$$

where $a_{1,k}$ and $b_{1,k}$ are constants. After solving Eq. (40) and Eq. (41) we have,

$$S_1(t) = -\frac{\hbar}{\alpha}\sum_{k=0}^{6}\frac{a_{1,k}}{k+1} e^{-\alpha(k+2)t} + \frac{\hbar}{\alpha}\left(\sum_{k=0}^{6}\frac{a_{1,k}}{k+1}\right)e^{-\alpha t}, \tag{42}$$

$$I_1(t) = -\frac{\hbar}{\alpha}\sum_{k=0}^{6}\frac{b_{1,k}}{k+1} e^{-\alpha(k+2)t} + \frac{\hbar}{\alpha}\left(\sum_{k=0}^{6}\frac{b_{1,k}}{k+1}\right)e^{-\alpha t}. \tag{43}$$

Similarly, for second order we have,

$$\begin{aligned}\Re_2^S(t) = &\sum_{k=0}^{6}\frac{a_{1,k}}{k+1}[(M_1 - (k+1)\frac{\hbar}{\alpha})e^{-(k+2)\alpha t} + M_2], \\ &+\sum_{k=0}^{6}\frac{b_{1,k}}{k+1}[M_3 e^{-(k+2)\alpha t} + M_4] - \mu S^0,\end{aligned} \tag{44}$$

$$\begin{aligned}\Re_2^I(t) = &\sum_{k=0}^{6}\frac{a_{1,k}}{k+1}[N_1 e^{-(k+2)\alpha t} + N_2], \\ &+\sum_{k=0}^{6}\frac{b_{1,k}}{k+1}[N_3 + h(k+1)e^{-(k+2)\alpha t} + N_4],\end{aligned} \tag{45}$$

where

M_1 [illegible]

M_2 [illegible]

M_3 [illegible]

M_4 [illegible]

N_1 [illegible]

N_2 [illegible]

N_3 [illegible]

N_4 [illegible]

with $M = S_0(t) - S_\infty$ and $N = I_0(t) - I_\infty$. Then the second-order deformation equations become,

$$S_2'(t) + \alpha S_2(t) = \hbar H_S(t) R_2^S(t), \quad S_2(0) = 0,$$

$$I_2'(t) + \alpha I_2(t) = \hbar H_I(t) R_2^I(t), \quad I_2(0) = 0,$$

After solving Eq. (Sec0.Eq39) and Eq. (Sec0.Eq40) we have,

$$S_2(t) = \hbar e^{-\alpha t}\int_0^t H_S(t)R_2^S(t)e^{\alpha t}dt, \tag{46}$$

$$I_2(t) = \hbar e^{-\alpha t}\int_0^t H_I(t)R_2^I(t)e^{\alpha t}dt, \tag{47}$$

In a similar way, it is easy to get $S_3(t)$, $I_3(t)$, $S_4(t)$, $I_4(t)$ and so on, especially by means of symbolic computation software such as Matlab, Mathematica and Maple. For th-order we have,

$$S_m(t) = \hbar e^{-\alpha t}\int_0^t H_S(t)R_m^S(t)e^{\alpha t}dt, \tag{48}$$

$$I_m(t) = \hbar e^{-\alpha t}\int_0^t H_I(t)R_m^I(t)e^{\alpha t}dt, \tag{49}$$

Results:

Example. In this example we used the following parameters, $S(0) = 120, I(0) = 40, S^0 = 2$, μ=0.1, $r\beta$=0.5, α = 8, For this example we have, R_0=5>1. According to the curve $S\sim\hbar$ at the th-order of approximation, the homotopy analysis method series are convergent in the region -1≤ħ≤-0.25 (See Fig.1). So, we choose ħ=-0.75, and the corresponding homotopy analysis method series converge to the numerical ones, as shown in Fig.2.

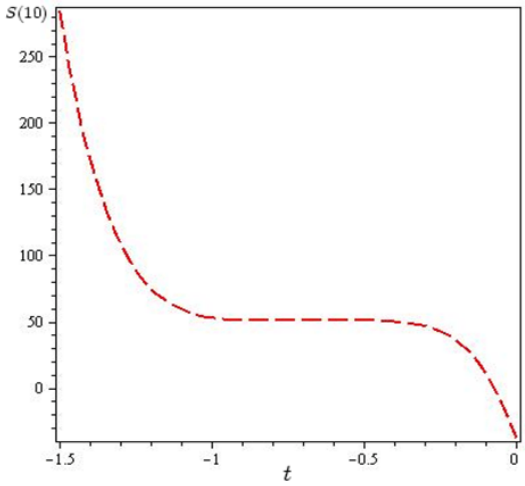

Fig. 1: The ħ-curve of 20th-order approximation by HAM for t=10.

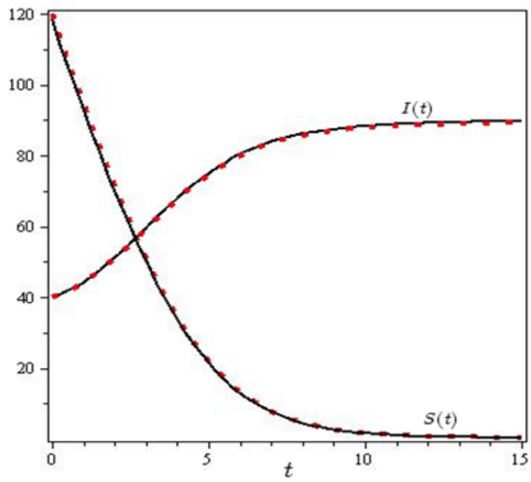

Fig. 2: Solid line: The susceptible and infective curves of 20th-order approximation by HAM when $\hbar$=-0.75, α=8, $H_S(t)$=$H_I(t)$ = e^{-at}, and dot line: the susceptible and infective curves of RKM5

Example. In this example we used the following parameters, $S(0)=120$, $I(0)=80$, $S^0=10$, $\mu=0.22$, $r\beta=0.2$, $\alpha = 3$, For this example we have, $R_0=0.91<1$. According to the curve $S \sim \hbar$ at the 20th-order of approximation, the homotopy analysis method series are convergent in the region $-1.5 \leq \hbar \leq 0$. So, we choose $\hbar=-1.25$, and the corresponding homotopy analysis method series converge to the numerical ones, as shown in Fig.3.

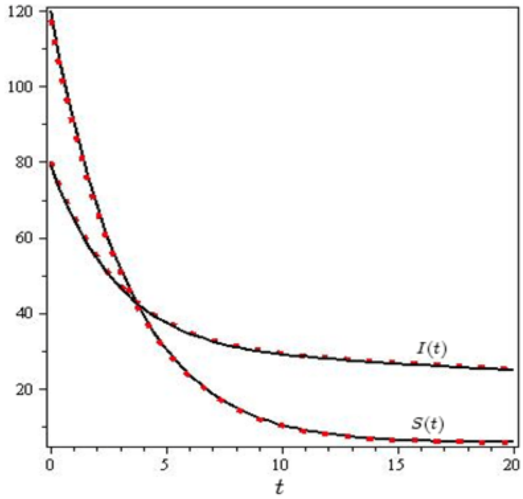

Fig. 3: Solid line: The susceptible and infective curves of 20th-order approximation by HAM when $\hbar=-1.25$, $\alpha=3$, $H_S(t)=H_I(t) = e^{-at}$, and dot line: the susceptible and infective curves of RKM5

Conclusions:

In this article we used the homotopy analysis method to find the exact analytical approximation for deterministic systems for susceptibe-infective model in the epidemic diseases. We obtained analytical approximation with high accuracy by taking suitable initial conditions.

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