Australian Journal of Basic and Applied Sciences, 5(10): 406-416, 2011
ISSN 1991-8178

The Modified Variational Iteration Method for Solving Linear and Nonlinear Ordinary Differential Equations

[1]M.A. Fariborzi Araghi, [2]S. Gholizadeh Dogaheh, [3]Z. Sayadi

[1]Department of Mathematics, Islamic Azad University, Central Tehran Branch, P.O.Box 13185.768, Tehran, Iran.
[2]Department of Mathematical Sciences, Islamic Azad University, Kermanshah Branch, Kermanshah, Iran.
[3]Department of Mathematical Sciences, University of Applied Science and Technology, Kermanshah Branch, Kermanshah, Iran.

Abstract: In this work, the modified variational iteration method (MVIM) is applied to solve linear and nonlinear ordinary differential equations such as Lane-Emden, Emden-Fowler and Riccati equations. The MVIM provides a sequence of functions which is convergent to the exact solution and is capable to cancel some of the repeated calculations and reduce the cost of operation in comparison with VIM. The method is very simple and easy.

Key words: modified variational iteration method, ordinary differential equations, Lane-Emden equation, Emden-Fowler equation, Riccati equation.

INTRODUCTION

The variational iteration method was first proposed by Ji-Huan He to find the solution of a differential equation using an iterative scheme (He J.H., 1998; 2000; 2006; 2007; 2008). Many researches in variety of scientific fields applied this method and showed the VIM has many merits and to be reliable for a variety of scientific application, linear and nonlinear as well (Abdou M.A. 2005; Abulwafa, 2006; S. Momani, 2005; 2006; Wazwaz, 2007; 2008).

Insight into the solution procedure of the VIM shows some disadvantages, namely, repeated computation of redundant terms, which wastes time and effort. Abassy *et al*., proposed the modified variational iteration method and used it to give an approximate power series solutions for some well-known nonlinear problems (Abassy, 2007).

The modified variational iteration method (MVIM) facilitates the computational work and minimizes it. This method can effectively improve the speed of convergence (Abassy, 2007).

In this work, we aim to show the power of MVIM in handling various types of ODEs of distinct orders. In fact this paper is an extension of the work done in (Wazwaz, 2009) which shows a new application of MVIM for linear and nonlinear homogeneous and inhomogeneous ODEs.

2. First Order ODEs:

First, we consider the first order linear ODE of a standard form

$$u' + p(x)u = q(x), \quad u(0) = \alpha. \tag{1}$$

According to the VIM, the basic character of the method is to construct a correction functional for the equation, which reads (Wazwaz, 2009):

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(u_n'(t) + p(t)\tilde{u}_n(t) - q(t))dt, \tag{2}$$

where λ is called a general Lagrange multiplier (M. Inokuti, 1978), which can be identified optimally via variational theory, $\tilde{u}_n$ denotes a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

Calculating variation with respect to u_n, the following stationary conditions are obtained (Wazwaz, 2009).

$$1 + \lambda\big|_{t=x} = 0\,, \quad \lambda'\big|_{t=x} = 0. \tag{3}$$

The Lagrange multiplier, therefore, can be identified as $\lambda = -1$.

By Substituting the identified multiplier into Eq. (2) the following iteration formula can be obtained as:

Corresponding Author: S. Gholizadeh Dogaheh, Department of Mathematical Sciences, Islamic Azad University, Kermanshah Branch, Kermanshah, Iran.
E-mail: gholizadeh_somaye@yahoo.com

$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(t) + p(t)u_n(t) - q(t))dt. \quad (4)$$

Recall that: $u(x) = lim_{n\to\infty} u_n(x)$.

In this method, there are repeated calculation in each step, to cancel some of the repeated calculation, the iteration formula (4) can be handled as follows

$$u_{n+1}(x) = u_n(x) - \int_0^x u_n'(t)\,dt - \int_0^x \big(p(t)u_n(t) - q(t)\big)\,dt.$$

Therefore

$$u_{n+1}(x) = u_n(0) - \int_0^x (p(t)u_n(t) - q(t))\,dt. \quad (5)$$

We can set $u_n(0) = u_0 = \alpha$, so

$$u_{n+1}(x) = u_0 - \int_0^x (p(t)u_n(t) - q(t))\,dt. \quad (6)$$

Via the iteration formula (6) some repeated computation are cancelled.
To eliminate all repeated computation, let us rewrite Eq. (6) in the following iteration formula:

$$u_{n+1}(x) = u_0 - \int_0^x p(t)(u_n(t) - u_{n-1}(t))\,dt - \int_0^x \big(p(t)u_{n-1}(t) - q(t)\big)\,dt. \quad (7)$$

But it is known from (6) that

$$u_n(x) = u_0 - \int_0^x (p(t)u_{n-1}(t) - q(t))\,dt. \quad (8)$$

Substituting by (8) in (7), we obtain

$$u_{n+1}(x) = u_n - \int_0^x p(t)(u_n(t) - u_{n-1}(t))dt\,,\ \ n > 0 \quad (9)$$

where $u_{-1} = 0$, $u_0 = u(0) = \alpha$ and u_1 is obtained from

$$u_1 = u_0 - \int_0^x (p(t)\big(u_0(t) - u_{-1}(t)\big) - q(t))dt.$$

This final modified formula (9) cancels all the repeated calculation and terms, which are not needed.
Now, we apply the MVIM for solving first order ODEs. Examples 2 and 3 are two well-known first order nonlinear equations, namely the logistic differential equation and the Riccati equation.

Notice that for nonlinear problems, the MVIM is not require specific treatment and approaches in a like manner to that used for linear problems.

Example1:

Now we consider the following first order inhomogeneous ODE

$$u' - u = e^x\ , \qquad u(0) = 0 \quad (10)$$

For solving this equation by MVIM, follow the discussion presented above we can set $u_0 = 0$, $u_{-1} = 0$ and we use the follow iteration formula

$$u_{n+1} = u_n - \int_0^x (-(u_n - u_{n-1}))\,dt\,, n > 0 \quad (11)$$

where $u_1 = u_0 - \int_0^x (-(u_0 - u_{-1}) - e^t)\,dt.$

Therefore by the above iteration formula, we can obtain following approximations

$$u_1 = u_0 - \int_0^x (-(u_0 - u_{-1}) - e^t)\,dt = e^x - 1,$$

$$u_2 = u_1 - \int_0^x \big(-(u_1 - u_0)\big)\, dt = 2e^x - x - 2 = 2\left(1 + x + \frac{x^2}{2!}\right) - x - 2 = x(1 + x),$$

$$u_3 = u_2 - \int_0^x \big(-(u_2 - u_1)\big)\, dt = 3e^x - 2x - \frac{1}{2}x^2 - 3$$

$$= 3\left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!}\right) - 2x - \tfrac{1}{2}x^2 - 3 = x\left(1 + x + \tfrac{x^2}{2!}\right),$$

$$u_4 = u_3 - \int_0^x \big(-(u_3 - u_2)\big)\, dt = 4e^x - 3x - x^2 - \frac{1}{6}x^3 - 4$$

$$= 4\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) - 3x - x^2 - \frac{1}{6}x^3 - 4 = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$\vdots$$

$$u_n(x) = x\left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!}\right).$$

Obtained by using the Taylor series for the obtained approximation. Recall that the exact solution can be obtained by using: $u(x) = \lim_{n\to\infty} u_n(x) = xe^x$, which are the same solutions as obtained by Wazwaz (2009) with VIM, but in this method as it is shown, the source inhomogeneous term is only used for the first iteration step (u_1) and this method reduces the size of calculation in comparison with VIM.

Example2:

Now we solve the follow nonlinear logistic differential equation by the MVIM:

$$u' = \mu u(1 - u)\ ,\quad u(0) = \tfrac{1}{2} \tag{12}$$

where $\mu > 0$ is a positive constant.

Follow the discussion presented above, we use the iteration formula

$$u_{n+1} = u_n - \textstyle\int_0^x -\mu(u_n - u_{n-1})\big(1 - (u_n - u_{n-1})\big)\, dt, n \geq 0. \tag{13}$$

Starting with initial approximation $u_0 = \frac{1}{2}$ and $u_{-1} = 0$ and by the iteration formula (13), we can obtain the following approximation:

$$u_1 = u_0 - \textstyle\int_0^x -\mu(u_0 - u_{-1})\big(1 - (u_0 - u_{-1})\big)\, dt = \frac{1}{2} + \frac{\mu}{4}x,$$

$$u_2 = u_1 - \textstyle\int_0^x -\mu(u_1 - u_0)\big(1 - (u_1 - u_0)\big)\, dt = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3,$$

$$u_3 = u_2 - \int_0^x -\mu(u_2 - u_1)\big(1 - (u_2 - u_1)\big)\, dt = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{\mu^7}{16128}x^7,$$

$$u_4 = u_3 - \textstyle\int_0^x -\mu(u_3 - u_2)\big(1 - (u_3 - u_2)\big)\, dt = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 + \frac{19\mu^9}{1451520}x^9,$$

$$u_5 = u_4 - \textstyle\int_0^x -\mu(u_4 - u_3)\big(1 - (u_4 - u_3)\big)\, dt = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 + \frac{31\mu^9}{1451520}x^9 + \cdots$$

This will yield the exact solution: $u(x) = lim_{n\to\infty} u_n(x) = \frac{e^{\mu x}}{1+e^{\mu x}}$, as obtained by Wazwaz (2009) using VIM.

Example 3:

We apply MVIM to solve the following Riccati equation.

$$u' = u^2 - 2xu + x^2 + 1\ ,\quad u(0) = 1. \tag{14}$$

Following the discussion presented above, we use the follow iteration formula

$$u_{n+1} = u_n - \int_0^x (-(u_n^2 - u_{n-1}^2) + 2t(u_n - u_{n-1}))dt, n > 0 \quad (15)$$

where $u_{-1} = 0$, $u_0 = 1$ and

$$u_1 = u_0 - \int_0^x (-(u_0^2 - u_{-1}^2) + 2t(u_0 - u_{-1}) - t^2 - 1)dt.$$

Therefore we can obtain the following successive approximation:

$$u_1 = u_0 - \int_0^x (-(u_0^2 - u_{-1}^2) + 2t(u_0 - u_{-1}) - t^2 - 1)dt = 1 + 2x + x^2 - \frac{1}{3}x^3,$$

$$u_2 = u_1 - \int_0^x (-(u_1^2 - u_0^2) + 2t(u_1 - u_0))dt = 1 + 2x + x^2 + \frac{1}{3}x^3 - \frac{1}{3}x^4 - \frac{1}{3}x^5 - \frac{1}{9}x^6 + \frac{1}{63}x^7,$$

$$u_3 = u_2 - \int_0^x (-(u_2^2 - u_1^2) + 2t(u_2 - u_1))dt = 1 + 2x + x^2 + x^3 + \frac{1}{3}x^4 - \frac{1}{15}x^5 - \frac{1}{9}x^6 + \cdots$$

$$u_4 = u_3 - \int_0^x (-(u_3^2 - u_2^2) + 2t(u_3 - u_2))dt = 1 + 2x + x^2 + x^3 + x^4 - \frac{11}{15}x^5 + \frac{19}{45}x^6 + \cdots$$

$$u_5 = u_4 - \int_0^x (-(u_4^2 - u_3^2) + 2t(u_4 - u_3))dt = 1 + 2x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots$$

$$\vdots$$

$$u_n(x) = x + (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \cdots).$$

This will yield the exact solution: $u(x) = x + \frac{1}{1-x}$, $|x| < 1$

3. Second Order ODEs:

We now consider the second order linear ODE with constant coefficients and extend our analysis to this equation that given by

$$u''(x) + au'(x) + bu(x) = g(x),\ u(0) = \alpha\ ,u'(0) = \beta \quad (16)$$

According to the VIM, the basic character of the method is to construct a correction functional for the equation, which reads (Wazwaz, 2009).

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(u_n''(t) + a\tilde{u}_n'(t) + b\tilde{u}_n(t) - g(t))dt, \quad n \geq 0 \quad (17)$$

Calculating variation with respect to u_n yields the following stationary conditions (Wazwaz, 2009).

$$1 - \lambda'\big|_{t=x} = 0 \quad , \quad \lambda(t)\big|_{t=x} = 0 \ , \ \lambda''(t)\big|_{t=x} = 0\,. \quad (18)$$

The Lagrange multiplier, therefore, can be identified as: $\lambda(t) = t - x$.

Substituting this value of the Lagrange multiplier into the functional (17) gives the iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (t - x)(u_n''(t) + au_n'(t) + bu_n(t) - g(t))dt\ ,\ n \geq 0 \quad (19)$$

It is observed that there are repeated calculations in each step. To stop these repeats, the following modification on the recursive formula (19) is suggested

$$u_{n+1}(x) = u_n(x) + \int_0^x (t - x)u_n''(t)dt + \int_0^x (t - x)(au_n'(t) + bu_n(t) - g(t))dt. \quad (20)$$

Integrating the integral by parts gives

$$\int_0^x (t-x)u_n''(t)dt = -u_n(x) + u_n(0) + xu_n'(0). \quad (21)$$

Substituting (21) into the (20) gives

$$u_{n+1}(x) = u_n(0) + xu_n'(0) + \int_0^x (t-x)(au_n'(t) + bu_n(t) - g(t))dt. \quad (22)$$

We can set $u_n(0) + xu_n'(0) = \alpha + \beta x = u_0$, so

$$u_{n+1}(x) = u_0 + \int_0^x (t-x)(au_n'(t) + bu_n(t) - g(t))dt. \quad (23)$$

To eliminate all the unneeded terms and the repeated computation in VIM we rewrite Eq. (23) in the following iteration formula

$$u_{n+1}(x) = u_0 + \int_0^x (t-x)(a(u_n'(t) - u_{n-1}'(t)) + b(u_n(t) - u_{n-1}(t))dt$$
$$+ \int_0^x (t-x)(au_{n-1}'(t) + bu_{n-1}(t) - g(t))dt. \quad (24)$$

But it is known from (23) that

$$u_n(x) = u_0 + \int_0^x (t-x)(au_{n-1}'(t) + bu_{n-1}(t) - g(t))dt. \quad (25)$$

So the following MVIM is used

$$u_{n+1}(x) = u_n + \int_0^x (t-x)(a(u_n'(t) - u_{n-1}'(t)) + b(u_n(t) - u_{n-1}(t))dt, \ \ n > 0 \quad (26)$$

where $u_0 = \alpha + \beta x$, $u_{-1} = 0$ and

$$u_1(x) = u_0 + \int_0^x (t-x)(a(u_0'(t) - u_{-1}'(t)) + b(u_0(t) - u_{-1}(t)) - g(t))dt.$$

Notice that the final modified formula (26) cancels all the repeated calculation and unsettled terms in VIM.

Now we apply the MVIM for solving two models of the second order ODEs, example 2 is a second order Euler equation.

Example1:

Solve the following second order inhomogeneous ODE.

$$u'' - 3u' + 2u = 2x - 3, \ \ u(0) = 1, \ \ u'(0) = 2. \quad (27)$$

For solving this equation by MVIM, follow the discussion presented above we can set $u_{-1} = 0$, $u_0 = 1 + 2x$. the iteration formula is given by

$$u_{n+1} = u_n + \int_0^x (t-x)\left(-3(u_n' - u_{n-1}') + 2(u_n - u_{n-1})\right)dt, n > 0 \quad (28)$$

where $u_1 = u_0 + \int_0^x (t-x)(-3(u_0' - u_{-1}') + 2(u_0 - u_{-1}) - 2t + 3)dt.$

Therefore we can obtain the following successive approximation:

$$u_0(x) = 1 + 2x,$$

$$u_1 = u_0 + \int_0^x (t-x)(-3(u_0' - u_{-1}') + 2(u_0 - u_{-1}) - 2t + 3)dt = 1 + 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3,$$

$$u_2 = u_1 + \int_0^x (t-x)\left(-3(u_1' - u_0') + 2(u_1 - u_0)\right)dt = 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 - \frac{1}{3}x^4 + \frac{1}{30}x^5,$$

$$u_3 = u_2 + \int_0^x (t-x)\left(-3(u_2' - u_1') + 2(u_2 - u_1)\right)dt,$$

$$= 1 + 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{13}{60}x^5 + \frac{7}{180}x^6 - \frac{1}{630}x^7,$$

$$\vdots$$

$$u_n(x) = x + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right).$$

This will yield the exact solution: $u(x) = lim_{n\to\infty} u_n(x) = x + e^x$, as obtained by Wazwaz (2009) using VIM.

Example 2:

We apply the MVIM to solve the following second order Euler equation

$$x^2y'' - 2xy' + 2y = 0 \quad , \; y(1) = 2, \quad y'(1) = 3 \; , \qquad x > 0 \tag{29}$$

Because of the singularity at x=0, we use the following transformation

$$z = ln\, x \;\Rightarrow\; x = e^z \tag{30}$$

So that

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{x}\frac{dy}{dz}, \\ \frac{d^2y}{dx^2} &= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}. \end{aligned} \tag{31}$$

Using (30) and (31) into (29) gives

$$\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 0, y(z=0) = 2 \, , \quad y'(z=0) = 3. \tag{32}$$

To solve Eq. (32) by the MVIM, follow the discussion presented above we can set $y_{-1} = 0$, $y_0 = 2 + 3z$. Then we use the iteration formula

$$y_{n+1}(z) = y_n(z) + \int_0^z (t - z)\left(-3(y_n' - y_{n-1}') + 2(y_n - y_{n-1})\right) dt \, , n \geq 0. \tag{33}$$

Therefore by the above iteration formula, we can obtain following approximations

$$y_1(z) = y_0(z) + \int_0^z (t - z)\left(-3(y_0' - y_{-1}') + 2(y_0 - y_{-1})\right) dt = 2 + 3z + \frac{5}{2}z^2 - z^3$$

$$= \left(1 + z + \frac{z^2}{2!}\right) + (1 + 2z + 2z^2) - z^3,$$

$$y_2(z) = y_1(z) + \int_0^z (t - z)\left(-3(y_1' - y_0') + 2(y_1 - y_0)\right) dt = 2 + 3z + \frac{5}{2}z^2 + \frac{3}{2}z^3 - \frac{7}{6}z^4 + \frac{1}{10}z^5$$

$$= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!}\right) + \left(1 + 2z + 2z^2 + \frac{8}{3!}z^3\right) + \cdots$$

$$y_3(z) = y_2(z) + \int_0^z (t - z)\left(-3(y_2' - y_1') + 2(y_2 - y_1)\right) dt$$

$$= 2 + 3z + \frac{5}{2}z^2 + \frac{3}{2}z^3 + \frac{17}{24}z^4 - \frac{17}{20}z^5 + \frac{23}{180}z^6 - \frac{1}{210}z^7$$

$$= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}\right) + \left(1 + 2z + 2z^2 + \frac{8}{3!}z^3 + \frac{16}{4!}z^4\right) + \cdots$$

$$y_4(z) = y_3(z) + \int_0^z (t - z)\left(-3(y_3' - y_2') + 2(y_3 - y_2)\right) dt$$

$$= 2 + 3z + \frac{5}{2}z^2 + \frac{3}{2}z^3 + \frac{17}{24}z^4 + \frac{11}{40}z^5 - \frac{17}{36}z^6 + \frac{2}{21}z^7 - \frac{2}{315}z^8 + \frac{1}{7560}z^9$$

$$= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\right) + \left(1 + 2z + 2z^2 + \frac{8}{3!}z^3 + \frac{16}{4!}z^4 + \frac{32}{5!}z^5\right) + \cdots$$

The last approximation convergences to the exact solution: $y(z) = \lim_{n\to\infty} y_n(x) = e^z + e^{2z}$.

Recall that $z = ln\, x$, then this will yield the exact solution: $y(x) = x + x^2$, which are the same solution as obtained by Wazwaz (2009) with VIM, but in this method as it is shown, reduces the size of calculation in comparison with VIM.

4. Third Order ODEs:

In this section, we consider the third order linear ODE with constant coefficients and apply the MVIM for solving this equation that given by

$$u''' + au'' + bu' + cu = g(x),\ u(0) = \alpha,\quad u'(0) = \beta,\quad u''(0) = \gamma. \tag{34}$$

According to the VIM, the basic character of the method is to construct a correction functional for the equation, which reads (Wazwaz, 2009).

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)(u_n'''(t) + a\tilde{u}_n''(t) + b\tilde{u}_n'(t) + c\tilde{u}_n(t) - g(t))dt\ ,\ n \geq 0 \tag{35}$$

We take the variation of both sides of (35) with respect to the independent variable u_n and notice that $\delta\tilde{u}_n = \delta\tilde{u}_n' = \delta\tilde{u}_n'' = 0$, Then we find

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\left(\int_0^x \lambda(t)u_n''(t)dt\right). \tag{36}$$

By taking the integral of (36) by parts three times and by using the extremum condition of u_{n+1} the following stationary conditions yields: $\lambda''' = 0$, $1 + \lambda''\big|_{t=x} = 0$, $\lambda(t)\big|_{t=x} = 0$.

The Lagrange multiplier, therefore, can be identified as: $\lambda(t) = \frac{-1}{2!}(t-x)^2$.

Substituting this value of the Lagrange multiplier into the functional (35) and delete the restriction on u_n, u_n', u_n'' this leads to:

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x (t-x)^2(u_n'''(t) + au_n''(t) + bu_n'(t) + cu_n(t) - g(t))dt. \tag{37}$$

In this method there are repeated calculation in each step, to cancel some of the repeated calculation, the iteration formula (37) can be handled as follows.

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x (t-x)^2 u_n'''(t)dt - \frac{1}{2!}\int_0^x (t-x)^2\,(au_n''(t) + bu_n'(t) + cu_n(t) - g(t))dt \tag{38}$$

Therefore

$$u_{n+1}(x) = u_n(0) + xu_n'(0) - \frac{x^2}{2}u_n''(0) - \frac{1}{2!}\int_0^x (t-x)^2\left(au_n''(t) + bu_n'(t)dt + cu_n(t) - g(t)\right)dt. \tag{39}$$

We can set $u_n(0) + xu_n'(0) - \frac{x^2}{2}u_n''(0) = \alpha + \beta x + \gamma\frac{x^2}{2} = u_0$ so

$$u_{n+1}(x) = u_0(x) - \frac{1}{2!}\int_0^x (t-x)^2\left(au_n''(t) + bu_n'(t) + cu_n(t) - g(t)\right)dt. \tag{40}$$

Via the iteration formula (40) some repeated computation are cancelled.

To eliminate all repeated computation, let us rewrite Eq. (40) in the following iteration formula:

$$u_{n+1}(x) = u_0(x) - \frac{1}{2!}\int_0^x (t-x)^2\left(a(u_n''(t) - u_{n-1}''(t)\right) + b(u_n'(t) - u_{n-1}'(t)) + c(u_n(t) - u_{n-1}(t))dt$$
$$-\frac{1}{2!}\int_0^x (t-x)^2\left(au_{n-1}''(t) + bu_{n-1}'(t) + cu_{n-1}(t) - g(t)\right)dt. \tag{41}$$

But it is known from (40) that

$$u_n(x) = u_0(x) - \frac{1}{2!}\int_0^x (t-x)^2 \left(a u''_{n-1}(t) + b u'_{n-1}(t) + c u_{n-1}(t) - g(t)\right) dt. \quad (42)$$

Substituting by (42) in (41), we get

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x (t-x)^2 \,(a(\,(u''_n(t) - u''_{n-1}(t)) + b(u'_n(t) - u'_{n-1}(t + c(u_n(t) - u_{n-1}(t)))dt), \quad (43)$$

where $u_{-1} = 0$, $u_0 = \alpha + \beta x + \gamma \frac{x^2}{2}$ and u_1 is obtained from

$$u_1(x) = u_0(x) - \frac{1}{2!}\int_0^x (t-x)^2 \,(a(u''_0(t) - u''_{-1}(t)) + b(u'_0(t) - u'_{-1}(t)) + c(u_0(t) - u_{-1}(t)) - g(t))dt.$$

This final modified formula (43) cancels all the repeated calculation and terms, which are not needed. Now, we apply the MVIM for solving a third order ODE.

Example1:

we consider the following third order inhomogeneous ODE

$$u''' - 2u'' + u' = 1 \quad , u(0) = 0, u'(0) = 2, u''(0) = 2. \quad (44)$$

For solving this equation by MVIM, follow the discussion presented above we can set $u_{-1} = 0$ and $u_0 = 2x + x^2$, then the following iteration formula obtain

$$u_{n+1} = u_n - \frac{1}{2}\int_0^x (t-x)^2 \left(-2(u''_n - u''_{n-1}) + (u'_n - u'_{n-1})\right) dt, n > 0 \quad (45)$$

where $u_1 = u_0 - \frac{1}{2}\int_0^x (t-x)^2 \,(-2(u''_0 - u''_{-1}) + (u'_0 - u'_{-1}) - 1)dt$

Therefore, we can obtain the following successive approximation:

$u_0 = 2x + x^2$,

$u_1 = u_0 - \frac{1}{2}\int_0^x (t-x)^2 \,(-2(u''_0 - u''_{-1}) + (u'_0 - u'_{-1}) - 1)dt = 2x + x^2 + \frac{1}{2}x^3 - \frac{1}{12}x^4$,

$u_2 = u_1 - \frac{1}{2}\int_0^x (t-x)^2 \left(-2(u''_1 - u''_0) + (u'_1 - u'_0)\right) dt = 2x + x^2 + \frac{1}{2}x^3 - \frac{1}{3!}x^4 - \frac{7}{120}x^5 + \frac{1}{360}x^6$,

$u_3 = u_2 - \frac{1}{2}\int_0^x (t-x)^2 \left(-2(u''_2 - u''_1) + (u'_2 - u'_1)\right) dt$

$= 2x + x^2 + \frac{1}{2}x^3 - \frac{1}{3!}x^4 + \frac{1}{4!}x^5 - \frac{1}{40}x^6 + \frac{11}{5040}x^7 - \frac{1}{20160}x^8$,

⋮

$u_n(x) = x + x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)$.

Consequently this will yield the exact solution: $u(x) = x(1 + e^x)$. In this method as it is shown, the source inhomogeneous term is only used for the first iteration step (u_1) and this method reduces the size of calculation in comparison with VIM (Wazwaz, 2009).

5. The Lane-Emden-Fowler Equation of Index M:

We close our study by using the MVIM to the Lane-Emden-Fowler equation of index m (X. Shang, 2009; Wazwaz, 2002; 2005) that given by

$$y''(x) + \frac{2}{x}y'(x) + af(x)y^m = 0 \quad , y(0) = 1 , \quad y'(0) = 0 \quad (46)$$

For $f(x) = 1$ and $a = 1$, Eq. (46) is reduced to the standard Lane-Emden equation. However, for $f(x) = x^n$, $a = 1$ Eq.(46) gives the Emden-Fowler equation of index *m*. for physical interest, the polytrophic index *m* lies between 0 and 5 (X. Shang, 2009) . Notice that this equation is only linear for $m = 0$ and $m = 1$. Otherwise it is nonlinear.

5.1. The Lane-Emden Equation of Index M:

The Lane-Emden equation of index m is given by

$$y''(x) + \frac{2}{x}y'(x) + y^m = 0\ , \quad y(0) = 1\ , \quad y'(0) = 0 \tag{47}$$

This equation was used to model the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical lows of thermodynamics.

Because of the singularity at *x*=0, we use the following transformation

$$u(x) = xy(x) \Rightarrow y(x) = \frac{u(x)}{x}. \tag{48}$$

So that

$$\begin{aligned} u'(x) &= xy' + y \Rightarrow y'(x) = \frac{u'(x)}{x} - \frac{u(x)}{x^2}\ , \\ u''(x) &= xy'' + 2y' \Rightarrow y''(x) = \frac{u''(x)}{x} - 2\frac{u'(x)}{x^2} + 2\frac{u(x)}{x^3}. \end{aligned} \tag{49}$$

Using (48) and (49) into (47) gives

$$u'' + x^{1-m}u^m = 0 \qquad m = 0,1,2,\ldots\ , \ u(0) = 0, \quad u'(0) = 1 \tag{50}$$

To solve this equation by The MVIM following the discussion presented before for second order equations with constant coefficients the following iteration formula is obtained:

$$u_{n+1}(x) = u_n(x) + \int_0^x (t-x)\left(t^{1-m}(u_n^m - u_{n-1}^m)\right)dt, n \geq 0 \tag{51}$$

where $u_{-1} = 0$, $u_0 = x$. Therefore we can obtained the following successive approximation

$u_0(x) = x,$

$$u_1(x) = u_0(x) + \int_0^x (t-x)\left(t^{1-m}(u_0^m - u_{-1}^m)\right)dt = x - \frac{1}{6}x^3,$$

$$u_2(x) = u_1(x) + \int_0^x (t-x)\left(t^{1-m}(u_1^m - u_0^m)\right)dt = x - \frac{1}{6}x^3 + \frac{m}{120}x^5,$$

$$u_3(x) = u_2(x) + \int_0^x (t-x)\left(t^{1-m}(u_2^m - u_1^m)\right)dt = x - \frac{x^3}{6} + \frac{m}{120}x^5 - \frac{m(8m-5)}{3(7!)}x^7,$$

$$u_4(x) = u_3(x) + \int_0^x (t-x)\left(t^{1-m}(u_3^m - u_2^m)\right)dt = x - \frac{x^3}{6} + \frac{m}{120}x^5 - \frac{m(8m-5)}{3(7!)}x^7 + \frac{m(70-183m+122m^2)}{9(9!)}x^9 + \cdots$$

Recall that $y(x) = \frac{u(x)}{x}$. This gives the series solution

$$y(x) = 1 - \frac{x^2}{6} + \frac{m}{120}x^4 - \frac{m(8m-5)}{3(7!)}x^6 + \frac{m(70-183\,m+122m^2)}{9(9!)}x^8 + \cdots \tag{52}$$

Exact Solutions exist only for three cases, namely:

Case (1): For $m = 0$, by substituting $m = 0$ in (52)the exact solution is given by

$$y(x) = 1 - \frac{x^2}{6},$$

Case (2): For $m = 1$, by substituting $m = 1$ in (52) the exact solution is given by

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{7!}x^6 + \frac{1}{9!}x^8 + \cdots = \frac{\sin x}{x},$$

Case (3): For $m = 5$, by substituting $m = 5$ in (52) the exact solution is given by

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{24}x^4 - \frac{5}{756}x^6 + \frac{5}{7776}x^8 + \cdots = \left(1 + \frac{x^2}{3}\right)^{\frac{-1}{2}},$$

Which are the same solutions as obtained by Shang (2009) and Wazwaz (2009). with VIM, but in this method the size of calculation reduces. For other cases, only the series solution are obtainable.

5.2. The Emden-Fowler Equation of Index M:

Many problem in mathematical physics and astrophysics can be modelled by the Emden-Fowler equation of index m (Shang, 2009; Wazwaz, 2002; 2005) that given by

$$y''(x) + \frac{2}{x}y'(x) + ax^n y^m = 0,\ y(0) = 1\ ,\ y'(0) = 0 \tag{53}$$

It will be shown by using the MVIM that exact solutions exist only for m=0,1and5.

Because of the singularity at *x*=0, we use the transformation (48) and (49) into (53) that gives

$$u'' + ax^{1+n-m}u^m = 0\ \ (m = 0,1,\dots), u(0) = 0\ ,\ u'(0) = 1. \tag{54}$$

For solving Eq. (54) by the MVIM, based on the discussion presented above for second order equations with constant coefficients, the following iteration formula is obtained

$$u_{n+1}(x) = u_n(x) + \int_0^x (t - x)\left(at^{1+n-m}(u_n^m - u_{n-1}^m)\right)dt, \tag{55}$$

where $u_{-1} = 0\ ,\ u_0 = x$.

This in turn gives the successive approximations

$$u_0 = x,$$

$$u_1(x) = u_0(x) + \int_0^x (t - x)\left(at^{1+n-m}(u_0^m - u_{-1}^m)\right)dt\ = x - \frac{a}{(3+n)(2+n)}x^{3+n},$$

$$u_2(x) = u_1(x) + \int_0^x (t - x)\left(at^{1+n-m}(u_1^m - u_0^m)\right)dt\ = x - \frac{a}{(n+3)(n+2)}x^{n+3} + \frac{a^2m}{2(2n+5)(n+3)(n+2)^2}x^{2n+5},$$

$$u_3(x) = u_2(x) + \int_0^x (t - x)\left(at^{1+n-m}(u_2^m - u_1^m)\right)dt$$

$$= x - \frac{a}{(n+3)(n+2)}x^{n+3} + \frac{a^2m}{2(2n+5)(n+3)(n+2)^2}x^{2n+5} - \frac{a^3m(8m+3mn-2n-5)}{6(3n+7)(2n+5)(n+3)^2(n+2)^3}x^{3n+7} + \cdots$$

Recall that $y(x) = \frac{u(x)}{x}$. This gives the series solution

$$y(x) = 1 - \frac{a}{(n+3)(n+2)}x^{n+2} + \frac{a^2m}{2(2n+5)(n+3)(n+2)^2}x^{2n+4} - \frac{a^3m(8m+3mn-2n-5)}{6(3n+7)(2n+5)(n+3)^2(n+2)^3}x^{3n+6} + \cdots \tag{56}$$

From the (56) we conclude that $n \neq -3, -2, -\frac{5}{2}, -\frac{7}{3}, -\frac{9}{4}, \dots$

Exact Solution Exist only for three cases, namely:

Case (1): For *m*=0, *n*=0, the exact solution is given by: $y(x) = 1 - \frac{a}{6}x^2$,

Case (2): For *m*=1, *n*=0, the exact solution is given by: $y(x) = \frac{\sin\sqrt{a}x}{\sqrt{a}\,x}$,

Case (3): For *m*=5, *n*=0, the exact solution is given by: $y(x) = \left(1 + \frac{ax^2}{3}\right)^{\frac{-1}{2}}$,

which are the same solutions as obtained by Shang (2009) and Wazwaz, (2009) but in this method the size of calculation reduces.

Conclusion :

The modified variational iteration method is remarkably effective for solving various types of ODEs of distinct orders. In this work, we employed the modified variational iteration method to investigate linear and nonlinear ordinary differential equations. This method is a very promoting method, which will be certainly found widely applications.

By analyzing the obtained results and procedures used in modified variational iteration method and variational iteration method we observed that the modified variational iteration method facilitates the computational work and minimizes it and this method cancels all the unsettled term in variational iteration method, also this method is faster than variational iteration method and save time.

REFERENCES

Abassy, T.A., M.A. El-Tawil and H. El-Zoheiry, 2007. Toward a modified variational iteration method. Journal of computational and Applied Mathematics, 207(1): 137-147.

Abassy T.A., M.A. El-Tawil and H. El-Zoheiry, 2007. Modified variational iteration method for Boussinesq equation. Computer & Mathematics with Applications, 54(7-8): 955-965.

Abdou, M.A. and A.A. Soliman, 2005. Variational iteration method for Solving Burger's equations. Journal of computational and Applied Mathematics, 181(2): 245-251.

Abulwafa, E.M., M.A. Abdou and A.A. Mahmoud, 2006. The solution of nonlinear coagulation problem with mass loss. Chaos Soliton Fract., 29: 313-330.

He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput .Appl. Mech. Eng., 167: 57-68.

He, J.H., 2000. variational iteration method for autonomous ordinary differential systems, Appl. Math. Compu., 114(2/3): 115-123.

He, J.H., 2006. Some asymptotic methods for strongly nonlinear equation, Int. J. Modern Phys. B20(10): 1141-1199.

He, J.H., 2007. variational iteration method-some recent results and new interpretations, Journal of Computational and Applied Mathematics., 207(1): 3-17.

He, J.H., X.h. Wu, 2007. Variational iteration method : new development and applications, Comput. Math. Appl., 54(7/8): 881-894.

He, J.H., 2008. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, Int. J. Modern Phys., B22(21): 3487-3578.

Inokuti, M., H. Sekine, T. Mura, 1978. General use of the Lagrange multiplier in nonlinear mathematical physics, in: S. Nemat-Nasser(Ed), Variational Method in the Mechanics of solids, Pergamon Press, New York, pp: 156-162.

Momani, S., S. Abusaad, 2005. Application of He's variational iteration method to Helmholtz equation , Chaos soliton Fract., 27(5): 1119-1123.

Momani, S., Z. Odibat, 2006. Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett., A1(53): 245-251.

Shang, X., P. Wu, X. Shao, 2009. An efficient method for solving Emden-Fowler equations, Journal of the Franklin Institute., 889-897.

Wazwaz, A.M., 2002. A new method for solving singular initial value problems in the second order differential equation, Applied Mathematics and Computation., 128: 47-57.

Wazwaz, A.M., 2002. Partial Differential Equation: Methods and Applications, Bakema Publhsher, The Netherlands.

Wazwaz, A.M., 2005. A new algorithm for solving differential equations of the Lane-Emden type (1999) Applied Mathematics and Computation., 161: 543-560.

Wazwaz, A.M., 2005. Adomian decomposition method for a reliable treatment of the Emden-Fowler equation, Applied Mathematics and Computation., 161: 543-560.

Wazwaz, A.M., 2007. The variational iteration method for rational solutions for KdV, K(2,2), Burgers and cubic Boussinesq equations, Journal of Computational and Applied Mathematics., 207: 18-23.

Wazwaz, A.M., 2007. A comparison between the variational iteration method and Adomian decomposition method, Journal of Computational and Applied Mathematics., 207: 129-136.

Wazwaz, A.M., 2007. The variational iteration method for exact solutions of Laplace equation, Phys. lett. A 363: 260-262 .

Wazwaz, A.M., 2007. The variational iteration method for solving linear and nonlinear systems of PDEs Comput., Appl. Math., 54: 895-902.

Wazwaz, A.M., 2007. The variational iteration method; a reliable tool for solving linear and nonlinear Wave equations, Comput. Appl. Math., 54: 926-932.

Wazwaz, A.M., 2007. The variational iteration method; a powerful scheme for handling linear and nonlinear diffusion equations, Comput. Math. Appl., 54: 1933-939.

Wazwaz, A.M., 2007. The variational iteration method for a reliable treatment of the linear and nonlinear Goursat problem, Applied Mathematics and Computation., 193: 455-462.

Wazwaz, A.M., 2008. A study on linear and nonlinear Schrodinger equations by the variational iteration method, Chaos soliton Fract., 37: 1136-1142.

Wazwaz, A.M., 2009. The variational iteration method for analytic treatment for linear and nonlinear ODEs, Applied Mathematics and Computation., 212: 120-134.